ON THE STRUCTURE OF FINITELY GENERATED DIFFERENTIAL MODULES OF EXTENSIONS OF VALUATION DOMAINS

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21. September 2023

1 Introduction

Motivated by the study of the valuation theory of deeply ramified fields [K-R] recently there has been some interest in describing the structure of the module $\Omega_{O_L|O_K}$ of Kähler differentials of an extension $O_L|O_K$ of valuation domains induced by a finite extension $(L|K,v)$ of valued fields – see [C-K], [C-K-R]. The obstacles in doing so of course are the fact that the valuation domains in general are non-noetherian and that the extension $O_L|O_K$ needs not be essentially of finite type. Consequently the »classical« theory of differentials is not sufficient to describe the structure of $\Omega_{O_L|O_K}$ but must be, so it seems, complemented by rather delicate limit arguments as in [Tha1], [Tha2] and [C-K]. In this context the modest goal of the present article is to describe the structure of $\Omega_{O_L|O_K}$ *assuming* that it is a finitely generated respectively finitely presented O_L-module and to relate the result to the general theory of modules over valuation domains.

The O_L-module $\Omega_{O_L|O_K}$ *is* finitely generated, even finitely presented, for essentially finitely generated extensions $O_L|O_K$, which however is a rather restrictive property as the subsequent result shows. The notation used is the same as in the article [C-K]. In particular: $vK \subseteq vL$ denote the value groups and $Kv \subseteq Lv$ the residue fields of v; moreover $K^h \subseteq L^h$ denote the henselizations with respect to v.

Theorem 1.1 ([C-N], Theorem 1.5, [Dat], Theorem 1.2): *For a finite extension of valued fields $(L|K,v)$ the following properties are equivalent:*

1. *The ring extension $O_L|O_K$ is essentially finitely generated.*
2. $[L^h : K^h] = (vL : vK)[Lv : Kv]$ *and* $(vL : vK) = \epsilon(vL|vK)$*, where*

$$\epsilon(vL|vK) := |\{\delta \in vL^{\geq 0} : \forall \gamma \in vK^{>0}\ \delta < \gamma\}|.$$

Note that the condition $(vL : vK) = \epsilon(vL|vK)$ appearing in Theorem 1.1 implies that the maximal ideals M_K and M_L of O_K and O_L are principal ideals if $(vL : vK) > 1$. This fact will be used frequently in the sequel.

In the concrete case of a *finite* extension $O_L|O_K$, which is covered by Theorem 1.1, the main result of the present article is:

THEOREM 1.2: *Let $(L|K, v)$ be a finite extension of valued fields for which the ring extension $O_L|O_K$ is finite. Let $p := \mathrm{char}(Kv)$ and $e := (vL : vK)$. A description of the structure of the differential module $\Omega_{O_L|O_K}$ is then given by the following list:*

1. *$e = 1$ and $Lv|Kv$ is separable: $\Omega_{O_L|O_K} = 0$.*
2. *$e > 1$ and $Lv|Kv$ is separable:*

$$\Omega_{O_L|O_K} = O_L\, dt \cong O_L/(x),$$

   where $M_L = tO_L$ and $vx = v(f'(t))$ for the minimal polynomial $f \in K^{\mathrm{ur}}[X]$ of t over the maximal unramified subextension $K^{\mathrm{ur}}|K^h$ of $L^h|K^h$.

   In particular: $vx = (e-1)vt$ if $p = 0$ or $p \nmid e$.

3. *$e = 1$ and $Lv|Kv$ is not separable:*

$$\Omega_{O_L|O_K} = O_L\, dt_1 \oplus \ldots \oplus O_L\, dt_m,$$

   where $t_1v, \ldots t_mv$ form a p-basis of $Lv|Kv$.

4. *$e > 1$ and $Lv|Kv$ is not separable:*

$$\Omega_{O_L|O_K} = O_L\, dt \oplus O_L\, dt_1 \oplus \ldots \oplus O_L\, dt_m,$$

   where the elements $t, t_1, \ldots, t_m$ are chosen as in points 2 and 3.

The summands $O_L\, dt_i$ appearing in points 3 and 4 are isomorphic to $O_L/(x_i)$ for certain $x_i \in O_L \setminus O_L^\times$.

If $L|K$ is separable, then $x \neq 0$ (point 2) and $x_i \neq 0$ for all i (points 3 and 4).

If $L|K$ is not separable, then in point 3 precisely $[L : K(L^p)]$ of the elements $x_1, \ldots, x_m$ are equal to 0. In point 4 the same holds for the elements $x, x_1, \ldots, x_m$.

2 Results on finitely generated differential modules $\Omega_{O_L|O_K}$

Let $(L|K, v)$ be an extension of valued fields and let $O_L|O_K$ be the induced extension of valuation domains. The Theorems 4.5 and 4.7 in [C-K] seem to indicate that the O_L-module $\Omega_{O_L|O_K}$ of Kähler differentials is rather rarely finitely generated. It is therefore worthwhile to investigate the implications this property has on the structure of the extension $(L|K, v)$.

Theorem 2.1: *Suppose that the differential module $\Omega_{O_L|O_K}$ of the extension $O_L|O_K$ of valuation domains is finitely generated and let $p := \mathrm{char}(Kv)$. Then*

- *in the case $p = 0$ the transcendence degree $\mathrm{trdeg}\,(Lv|Kv)$ is finite,*
- *in the case $p \neq 0$ the degree $[Lv : Kv(Lv^p)]$ is finite.*

Let $t_1, \dots, t_m \in O_L$ be elements such that either $t_1 v, \dots, t_m v$ is a transcendence basis of $Lv|Kv$ ($p = 0$) or is a p-basis of $Lv|Kv$ ($p \neq 0$). Then

1. *$\Omega_{O_L|O_K} = O_L\, dt_1 \oplus \dots \oplus O_L\, dt_m$ if $(vL : vK) = 1$ or $(vL : vK) > \epsilon(vL|vK)$,*
2. *$\Omega_{O_L|O_K} = O_L\, dt \oplus O_L\, dt_1 \oplus \dots \oplus O_L\, dt_m$, where $tO_L = M_L$, if $(vL : vK) = \epsilon(vL|vK) > 1$.*

Proof: For a separable residue field extension $Lv|Kv$ the sequence of Lv-vector spaces

$$0 \to M_L/(M_L^2 + M_K O_L) \to \Omega_{O_L|O_K}/M_L \Omega_{O_L|O_K} \xrightarrow{\psi} \Omega_{Lv|Kv} \to 0 \qquad (1)$$

is exact – see [Kun], Corollary 6.5. Consequently the dimension of $\Omega_{Lv|Kv}$ is finite, which implies that there exists a basis $d\bar{t}_1, \dots, d\bar{t}_m$ of $\Omega_{Lv|Kv}$. In the case $p = 0$ the elements $\bar{t}_1, \dots, \bar{t}_m$ then form a transcendence basis of $Lv|Kv$, which proves the first assertion of the theorem.

In the case of an inseparable residue field extension $Lv|Kv$ the sequence of Lv-vector spaces

$$0 \to M_L/(M_L^2 + M' O_L) \to \Omega_{O_L|O_K}/M_L \Omega_{O_L|O_K} \xrightarrow{\psi} \Omega_{Lv|Kv} \to 0, \qquad (2)$$

is exact, where $M' := M_L \cap O_K[O_L^p]$, – see [Kun], Theorem 6.7. Again this yields the existence of a basis $d\bar{t}_1, \dots, d\bar{t}_m$ of $\Omega_{Lv|Kv}$, but since $p \neq 0$ the elements $\bar{t}_1, \dots, \bar{t}_m$ this time form a p-basis of $Lv|Kv$, which proves the second assertion of the theorem.

The homomorphism ψ in the sequences (1) and (2) is defined by $x\,dy + M_L\Omega_{O_L|O_K} \mapsto xv\,dyv$. Therefore it is possible to lift the differentials $d\bar{t}_i$ to differentials $dt_i \in \Omega_{O_L|O_K}$; the family $dt_1, \ldots, dt_m$ then is linearly independent over O_L.

In the case $(vL : vK) = 1$ the equation $M_K O_L = M_L$ holds, which implies $M_L/(M_L^2 + M_K O_L) = 0$ and $M_L/(M_L^2 + M' O_L) = 0$ for $p \neq 0$. Consequently the exact sequences (1) and (2) yield

$$\Omega_{O_L|O_K} = O_L\,dt_1 \oplus \ldots \oplus O_L\,dt_m$$

by the assumption that $\Omega_{O_L|O_K}$ be finitely generated and Nakayama's lemma.

In the case $(vL : vK) > \epsilon(vL|vK)$ the maximal ideal M_L is not finitely generated, hence $M_L^2 = M_L$ holds, which implies $M_L/(M_L^2 + M_K O_L) = 0$ and $M_L/(M_L^2 + M' O_L) = 0$ for $p \neq 0$. Then the same reasoning as above applies. Altogether the assertion 1 of the theorem is proved.

Finally suppose $(vL : vK) = \epsilon(vL|vK) > 1$ holds. Then $M_L = tO_L$ for some $t \in O_L$, which shows that $M_L/(M_L^2 + M_K O_L)$ and $M_L/(M_L^2 + M' O_L)$ for $p \neq 0$ are generated by the respective residue class of t. The homomorphism ϕ is defined by

$$x + (M_L^2 + M_K O_L) \mapsto dx + M_L\Omega_{O_L|O_K}$$

and similarly in the inseparable case. Consequently the exact sequences (1) and (2) yield that

$$dt + M_L\Omega_{O_L|O_K}, dt_1 + M_L\Omega_{O_L|O_K}, \ldots, dt_m + M_L\Omega_{O_L|O_K}$$

form a basis of $\Omega_{O_L|O_K}/M_L\Omega_{O_L|O_K}$, which again by Nakayama's lemma implies the assertion 2 of the theorem. □

Given Theorem 2.1 in order to understand the structure of $\Omega_{O_L|O_K}$ it is necessary to determine the annihilators $\mathrm{Ann}\,(ds)$, $s \in \{t, t_1, \ldots, t_m\}$. In particular the number r among these annihilators equal to 0 determines a decomposition of $\Omega_{O_L|O_K}$ into its torsion submodule and a free complement. Such a decomposition exists in general for finitely generated modules over valuation domains:

Proposition 2.2: *Let N be a finitely generated module over the valuation domain O and let T be the submodule of torsion elements of N. Then $N = T \oplus F$, where $F \cong O^r$ for some $r \in \mathbb{N}_0$.*

Proof: Any finitely generated, torsion-free module N over O is free: choose preimages $b_1, \ldots, b_r \in N$ of a basis of the O/M-vector space N/MN, where M is the maximal ideal of O. Then $N = Ob_1 \oplus \ldots \oplus Ob_r$.

In general the factor module N/T is torsion-free hence free, which implies that the exact sequence $0 \to T \to N \to N/T \to 0$ splits. □

For differential modules $\Omega_{O_L|O_K}$ the rank r of the free component in Proposition 2.2 can be determined. For a large class of extensions $O_L|O_K$ it is equal to 0:

PROPOSITION 2.3: *For a separable, algebraic extension $(L|K, v)$ of valued fields the differential module $\Omega_{O_L|O_K}$ is a torsion module.*

PROOF: It suffices to show that all exact differentials dx are torsion elements. Let $f \in K[X]$ be the minimal polynomial of x over K. Then there exists $\lambda \in O_K \setminus 0$ such that $\lambda f \in O_K[X]$. The separability of x over K yields $\lambda f'(x) \neq 0$. The equation

$$\lambda f'(x)\, dx = d(\lambda f(x)) = 0$$

therefore shows that dx is a torsion element. □

It remains to consider the case of an inseparable extension $L|K$.

PROPOSITION 2.4: *Let $(L|K, v)$ be a valued field extension of characteristic $p \neq 0$ and suppose that the differential module $\Omega_{O_L|O_K}$ is finitely generated. If $\Omega_{O_L|O_K} = T \oplus F$, $F \cong O_L^r$ is the decomposition of $\Omega_{O_L|O_K}$ according to Proposition 2.2, then $p^r = [L : K(L^p)]$.*

PROOF: Localization with respect to $O_L \setminus 0$ yields

$$\Omega_{L|O_K} \cong_L \Omega_{O_L|O_K} \otimes_{O_L} L \cong_L (T \oplus O_L^r) \otimes_{O_L} L \cong_L L^r.$$

Now $\Omega_{L|O_K} = \Omega_{L|K}$ and by [Kun], Proposition 5.7 the differential module $\Omega_{L|K}$ is an L-vector space of dimension r, where $p^r = [L : K(L^p)]$. □

Theorem 2.1 shows that the O_L-module $\Omega_{O_L|O_K}$ is isomorphic to a module of the form

$$O_L/I_1 \oplus \ldots \oplus O_L/I_\ell, \tag{3}$$

where I_k are proper ideals of O_L. The following result of Salce and Zanardo shows, that the ideals I_k are uniquely determined by $\Omega_{O_L|O_K}$:

THEOREM 2.5 ([F-S], Ch.V, Theorem 5.5): *Let N be a finitely generated module over the valuation domain O and suppose that*

$$0 =: U_0 \subset U_1 \subset \ldots \subset U_\ell := N \text{ and } 0 =: V_0 \subset V_1 \subset \ldots \subset V_{\ell'} := N$$

are chains of pure submodules of N such that the modules U_{i+1}/U_i and V_{i+1}/V_i are generated by one element. Then $\ell = \ell'$ and there exists a permutation $\sigma \in S_\ell$ such that $V_{i+1}/V_i \cong U_{\sigma(i)+1}/U_{\sigma(i)}$ for all i. In particular the annihilators $\mathrm{Ann}\,(U_{i+1}/U_i)$ up to their order are independent of the particular chain of submodules.

Corollary 2.6: *The annihilator ideals* Ann (ds), $s \in \{t, t_1, \ldots, t_m\}$, *appearing in Theorem 2.1 up to their order do not dependent on the choice of the elements* $t, t_1, \ldots, t_m$.

Proof: Direct summands of a module are pure. Hence Theorem 2.1 gives a chain of pure submodules in $\Omega_{O_L|O_K}$ as in Theorem 2.5. The annihilator ideals of this chain are the ideals Ann (ds), $s \in \{t, t_1, \ldots, t_m\}$. □

3 Results on finitely presented differential modules $\Omega_{O_L|O_K}$

For an extension $O_L|O_K$ of valuation domains the differential module $\Omega_{O_L|O_K}$ is a finitely generated O_L-module provided that O_L is an essentially finitely generated O_K-algebra. In this case one actually gets a much stronger result that can simplify the computation of the annihilators of the differentials appearing in Theorem 2.1.

Proposition 3.1: *Suppose that the extension $O_L|O_K$ of valuation domains is essentially finitely generated. Then the differential module $\Omega_{O_L|O_K}$ is a finitely presented O_L-module.*

Proof: The ring O_L has the form $O_L = A_q$ for some prime ideal q of a finitely generated O_K-algebra $A \subseteq \mathrm{Frac}\,(O_L)$. It suffices to show that $\Omega_{A|O_K}$ is a finitely presented A-module. Every flat, finitely generated algebra over a valuation ring is finitely presented – see [Nag-1], Theorem 3. Moreover for modules over valuation domains flatness and being torsion-free are equivalent. Thus there exists a presentation

$$0 \to (f_1, \ldots, f_r) \to S \to A \to 0,$$

where $S := O_K[X_1, \ldots, X_m]$ is the polynomial ring in m indeterminates over O_K. The corresponding conormal sequence

$$(f_1, \ldots, f_r)/(f_1, \ldots, f_r)^2 \to \Omega_{S|O_K} \otimes_S A \to \Omega_{A|O_K} \to 0$$

is exact. Since $\Omega_{S|O_K}$ is a free S-module possessing the basis $dX_1, \ldots, dX_m$, the tensor product $\Omega_{S|O_K} \otimes_S A$ is a free A-module, thus proving the assertion. □

Recall that a module N over a commutative ring R is called coherent, if every finitely generated submodule of N is finitely presented. A commutative ring R is coherent, if it is coherent as an R-module. Valuation domains are coherent rings, since their finitely generated ideals are principal.

Corollary 3.2: *For every* $\omega \in \Omega_{O_L|O_K}$ *the annihilator* $\mathrm{Ann}(\omega)$ *is a principal ideal of* O_L.

Proof: Every finitely presented module over a coherent ring is coherent – see [Gla], Theorem 2.3.2. Proposition 3.1 thus yields that $\Omega_{O_L|O_K}$ is coherent. Consequently in the exact sequence

$$0 \to \mathrm{Ann}(\omega) \to O_L \to O_L\omega \to 0$$

the module $O_L\omega$ is finitely presented, therefore $\mathrm{Ann}(\omega)$ is finitely generated thus principal. □

As a consequence of this corollary and Theorem 2.1 a finitely presented differential module $\Omega_{O_L|O_K}$ is isomorphic to an O_L-module of the form

$$O_L/(x_1) \oplus \ldots \oplus O_L/(x_\ell),\ x_1, \ldots, x_\ell \in O_L \setminus O_L^\times.$$

This is in fact a special case of a structure theorem by R. B. Warfield:

Theorem 3.3 ([F-S], Ch. I, Theorem 7.9): *Let* N *be a finitely presented module over a valuation domain* O *and let* $\ell \in \mathbb{N}$ *be the minimal number of generators of* N*. Then there exist elements* $x_1, \ldots, x_\ell \in O \setminus O^\times$ *such that*

$$N \cong O/(x_1) \oplus \ldots \oplus O/(x_\ell).$$

The ideals $(x_1), \ldots, (x_\ell)$ *are uniquely determined by* N *up to their order.*

Remark: A principal ideal (x) of a valuation domain O with associated valuation $v : K \to vK \cup \infty$ is determined by the value vx. Moreover one can assume that the ideals (x_i) appearing in Theorem 3.3 form a descending chain. Then the assignment

$$O\text{-Mod}_{\mathrm{fp}} \to F(vL^{>0}),\ N \mapsto [N] := (vx_1, \ldots, vx_\ell), \tag{4}$$

where O-$\mathrm{Mod}_{\mathrm{fp}}$ is the class of finitely presented O-modules and

$$F(vL^{>0}) := \{(\gamma_1, \ldots, \gamma_\ell) : \ell \in \mathbb{N}, \gamma_i \in vK \cup \infty, 0 < \gamma_1 \leq \ldots \leq \gamma_\ell\} \tag{5}$$

maps isomorpism classes bijectively to $F(vL^{>0})$.

The invariants $[N]$ of a finitely presented module are invariant under base change to the henselization, which is of particular interest in the case of a differential module:

Theorem 3.4: *Let $O_L|O_K$ be a unibranched extension of valuation domains and let O_K^h and O_L^h be their respective henselizations. Then*

$$\Omega_{O_L^h|O_K^h} \cong \Omega_{O_L|O_K} \otimes_{O_K} O_K^h.$$

Consequently if

$$\Omega_{O_L|O_K} \cong O_L/(x_1) \oplus \ldots \oplus O_L/(x_\ell),$$

then

$$\Omega_{O_L^h|O_K^h} \cong O_L^h/(x_1) \oplus \ldots \oplus O_L^h/(x_\ell).$$

Proof: By [Nag-2], Theorem 43.17 the henselization O_L^h as an O_L-algebra is isomorphic to the tensor product $O_L \otimes_{O_K} O_K^h$, hence

$$\Omega_{O_L^h|O_K^h} \cong \Omega_{O_L \otimes_{O_K} O_K^h|O_K^h} \cong \Omega_{O_L|O_K} \otimes_{O_K} O_K^h.$$

Since the extension $O_K^h|O_K$ is flat, for $x \in O_L$ one has

$$O_L/xO_L \otimes_{O_K} O_K^h \cong O_L \otimes_{O_K} O_K^h/xO_L \otimes_{O_K} O_K^h \cong O_L^h/xO_L^h,$$

which proves the second assertion of the theorem. $\square$

Corollary 3.5: *Let $(L|K, v)$ be a finite extension of valued fields with the properties: $Lv|Kv$ is separable, $O_L|O_K$ is finite and $(vL : vK) > 1$. Then $\Omega_{O_L|O_K} \cong O_L/(x)$ with an $x \in O_L$ such that $vx = v(f'(t))$, where $tO_L = M_L$ and $f \in K^{\mathrm{ur}}[X]$ is the minimal polynomial of t over the maximal unramified subextension $K^{\mathrm{ur}}|K^h$ of $L^h|K^h$. (The unique extension of v to the henselization L^h is again denoted by v.)*

In particular in the case $\mathrm{char}(Kv) =: p = 0$ *or* $p \nmid (vL : vK) =: e$ *one has* $vx = (e-1)vt$.

Proof: By assumption $Lv = Kv(\overline{\theta})$ and the minimal polynomial $\overline{g}$ of $\overline{\theta}$ over Kv is separable. Let $g \in O_K^h[X]$ be a monic polynomial with the property $\deg(g) = \deg(\overline{g})$ and $gv = \overline{g}$. Then g is irreducible over K^h and by Hensel's Lemma has a simple root $\theta \in O_L^h$ with $\theta v = \overline{\theta}$. Consequently the field extension $K^h(\theta)|K^h$ is separable of degree $[K^h(\theta) : K^h] = [Lv : Kv]$.

By Theorem 1.1 the degree of the extension $L^h|K^h$ is given by

$$[L^h : K^h] = (vL : vK)[Lv : Kv],$$

hence $[L^h : K^h(\theta)] = (vL : vK) = \epsilon(vL|vK)$; in particular $K^{\mathrm{ur}} = K^h(\theta)$.

Since $(vL : vK) = (vL^h : vK^h)$ and $\epsilon(vL|vK) = \epsilon(vL^h|vK^h)$, for every $t \in O_L$ with $tO_L = M_L$ one gets $O_{L^h} = O_{K^{\mathrm{ur}}}[t]$, therefore

$$\Omega_{O_{L^h}|O_{K^{\mathrm{ur}}}} \cong O_{L^h}/(f'(t)),$$

where f' is the minimal polynomial of t over K^{ur}.

In the exact sequence

$$\Omega_{O_{K^{\mathrm{ur}}}|O_{K^h}} \otimes_{O_{K^{\mathrm{ur}}}} O_{L^h} \to \Omega_{O_{L^h}|O_{K^h}} \to \Omega_{O_{L^h}|O_{K^{\mathrm{ur}}}} \to 0$$

by Proposition 6.8 in [Kun] $\Omega_{O_{K^{\mathrm{ur}}}|O_{K^h}} = 0$, hence $\Omega_{O_{L^h}|O_{K^h}} \cong \Omega_{O_{L^h}|O_{K^{\mathrm{ur}}}}$. Theorem 3.4 now yields

$$\Omega_{O_L|O_K} \cong O_L/(x),\ vx = v(f'(t)).$$

If $p = 0$ or $p \nmid e$, then the extension $L^h|K^{\mathrm{ur}}$ and thus

$$f = X^e + a_{e-1}X^{e-1} + \ldots + a_1 X + a_0$$

are separable. Since K^{ur} is henselian the conjugates $\sigma(t)$, $\sigma : L^h \to \widetilde{K^h}$ a K^{ur}-embedding of L^h into the algebraic closure of K^h, all possess the same value. Therefore $v(a_i) \geq (e-i)v(t)$ for all i. Since $v(t)$ is the minimal positive value of vL this implies

$$\begin{aligned} v(f'(t)) &= \min(v(e) + (e-1)v(t), v(ia_i t^{i-1}) : i \in \{1, \ldots, e-1\}) \\ &= (e-1)v(t). \end{aligned}$$

□

The ingredients for the proof of Theorem 1.2 are now available:

- Assertion 1 follows from [Kun], Proposition 6.8.
- Assertion 2 is Corollary 3.5.
- Assertion 3 follows from point 1 of Theorem 2.1.
- Assertion 4 follows from point 2 of Theorem 2.1.
- The statement about the structure of the summands $O_L\, dt_i$ follows from Corollary 3.2.
- The statement about the number of zero-elements among $x, x_1, \ldots, x_m$ is a consequence of the Propositions 2.3 and 2.4.

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