III Martingales

Example (continued from last section)

coin-tossing

$\Omega_0 = \{H, T\}$

$\Omega = \{(\omega_k)_{k\in\mathbb{N}} \mid \omega_k \in \Omega_0\}$, $\mathcal{A} = \mathcal{P}(\Omega)$

$\mathcal{A}_n = \{A_1 \times \dots \times A_n \times \Omega_0 \times \Omega_0 \times \dots \mid A_j \subseteq \Omega_0 \text{ for } 1 \le j \le n\}$ sub σ-field

$\subseteq \mathcal{A}_n \subseteq \mathcal{A}_{n+1} \subseteq \dots$

$$S_n(\omega) = \sum_{k=1}^{n} \underbrace{\left(1_H(\omega_k) - 1_T(\omega_k)\right)}_{G_k(\omega_k)}$$

$$S = \lim_{n\to\infty} S_n$$

Let $n \le m$.

$$E(S_m \mid \mathcal{A}_n) = E\left(\sum_{k=1}^{m} G_k \mid \mathcal{A}_n\right)$$

$$= E(S_n \mid \mathcal{A}_n) + E\left(\sum_{k=n+1}^{m} S_k \mid \mathcal{A}_n\right)$$

$$= S_n + \sum_{k=n+1}^{m} \underbrace{E(S_k \mid \mathcal{A}_n)}_{=0 \text{ (independence)}}$$

because S_n is $\mathcal{A}_n$-measurable

$$= S_n$$

That is, $E(S_m \mid \mathcal{A}_n) = S_n$ if $n \le m$.

Definition

$(\Omega, \mathcal{A}, P)$ probability space

$(\mathcal{A}_n)_{n \in \mathbb{N}}$ family of sub σ-fields

$\mathcal{A}_n \subseteq \mathcal{A}_m$ if $n \leq m$

Such a family of sub σ-fields is called a filtration.

Definition

$(\Omega, \mathcal{A}, P)$ probability space

$(\mathcal{A}_n)_{n \in \mathbb{N}}$ filtration

$(X_n)_{n \in \mathbb{N}}$ family of random variables $\Omega \to \mathbb{R}$, $\in L^1$.

Each X_n is $\mathcal{A}_n$-measurable.

If
$$E(X_m \mid \mathcal{A}_n) = X_n \qquad \forall n \leq m,$$
then (X_n) is a called a martingale.

Martingales are a generic model for fair games.
Is it reasonable to stop gambling, for instance after you have already won some money?

Let T be the random variable that describes the stopping:

S_n = gain up to time n

$$S_n'(\omega) = \begin{cases} S_n(\omega) & , \text{ if } \omega \in \{n \leq T\} \\ S_{T(\omega)}(\omega) & , \text{ if } \omega \in \{n > T\} \end{cases}$$

The decision for stopping should only depend upon the information that is already known, that is, not upon future outcomes of the game.

Definition

$\Omega \neq \emptyset$

$(\mathcal{A}_n)_{n\in\mathbb{N}}$ filtration

$T: \Omega \to \mathbb{N}$ is called stopping time

if $\{T \leq n\} \in \mathcal{A}_n \quad \forall n$.

Does it improve the situation if the gamble stops or participates only selectively ("optional sampling")? No!

<u>Theorem</u> (optional sampling theorem)

$(X_i)_{i=1,\dots,n}$ martingale w.r.t. $(\mathcal{A}_i)_{i=1,\dots,n}$

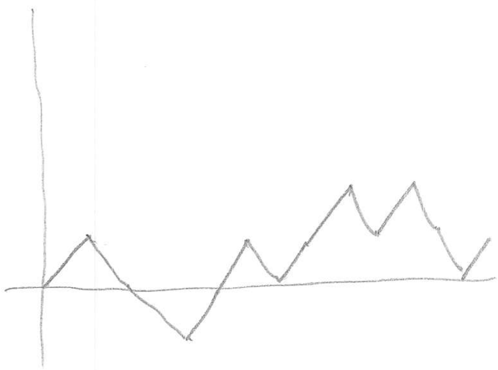
$(T_j)_{j=1,\dots,p}$ family of stopping times

with the property $T_j \leq T_k$ if $j \leq k$.

Then $(X_{T_j})_{j=1,\dots,p}$ is a martingale w.r.t. $(\mathcal{A}_{T_j})_{j=1,\dots,p}$

<u>Proof</u>: Bauer, probability theory (original proof by Doob)

Do martingales converge?



<u>Theorem</u>

(X_n) martingale

$\sup_{n\in\mathbb{N}} E(|X_n|) < \infty$

Then (X_n) converges almost surely to an integrable

random variable X_∞:

$$X_\infty = \lim_{n\to\infty} X_n \quad \text{almost surely}$$

<u>Proof</u>: Bauer

(X_n) martingale w.r.t. $(\mathcal{A}_n)$

Define $$\mathcal{A}_\infty := \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n\right)$$

σ-field generated by all $\mathcal{A}_n$

Theorem

Under a technical condition (uniform integrability)

$(X_n)_{n \in \mathbb{N} \cup \{\infty\}}$ is a martingale w.r.t. $(\mathcal{A}_n)_{n \in \mathbb{N} \cup \{\infty\}}$

Proof: Bauer

Theorem (martingale representation theorem)

(X_n) martingale w.r.t. $(\mathcal{A}_n)$, uniformly integrable

Then there is a random variable $X \in L_1$ such that

$$X_n = E(X \mid \mathcal{A}_n) \qquad \forall n \in \mathbb{N}$$

Proof

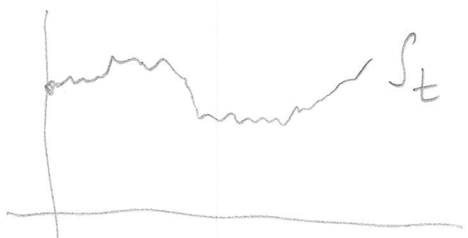
Take $X = \lim_{n \to \infty} X_n$ (see previous theorems).

Remark

Martingales and stopping times can be defined in a more general setup. The index set $\mathbb{N}$ can be replaced by $[0,\infty[$. This is important if we want to do mathematical finance in continuous time.

IV Application to mathematical finance

S_t price of a stock at time t



family of random variables indexed by t (a „stochastic process")

call option

fix $K>0$, T (maturity)

$C_T = \max(S_T - K, 0)$ payoff of the call option at maturity

European call option: can only be exercised at maturity T

American call option: can be exercised at any time prior to or equal to T.

More generally, a derivative is an instrument with a payoff of the form

$$X_T = f(S_T)$$

where f is a specified function and T is the maturity.

<u>Definition</u>

$(\Omega, \mathcal{A}, P)$ probability space

$(\mathcal{A}_t)$ filtration

(S_t) is called a primary security

if (S_t) is $(\mathcal{A}_t)$-adapted and $S_t > 0$ almost surely $\forall t$

typical primary securities:
stocks, government bonds, commodity prices (gold, silver...)

<u>Definition</u>

$S^1_t, \dots, S^n_t$ $\quad n$ primary securities

$a = (a^1_t, \dots, a^n_t)$ is called a trading strategy

if it is $(\mathcal{A}_t)$-adapted (i.e. trading decisions at time t depend only upon information up to time t, you cannot look into the future).

value of the portfolio at time t:

$$V_t(a) = \sum_{i=1}^{n} a_t^i S_t^i$$

cumulative gain:

$$G_t(a) = \sum_{i=1}^{n} \sum_{u=1}^{t} a_{u-1}^i (S_u^i - S_{u-1}^i)$$

<u>Definition</u>

A trading strategy a is called self-financing if

$$V_t(a) = V_0(a) + G_t(a) .$$

That is, you can only buy new securities by liquidating some of your existing positions (no injection or withdrawal of money).

<u>Definition</u>

A numeraire $N = (N_t)$ is any strictly positive value of a self-financing trading strategy

$$N_t = V_t(a) = V_0(a) + G_t(a) .$$

In particular, any primary security (stock, government bond, gold price) is a numeraire. It is the "currency" of measuring your wealth.

Definition

$(\Omega, \mathcal{A}, P)$ probability space

N numeraire

Q_N probability measure equivalent to P (i.e. with mutual densities)

Q_N is called equivalent martingale measure for the numeraire N if all primary securities expressed in numeraire units are Q_N-martingales, i.e.

$$\left(\frac{S_t^i}{N_t}\right) \quad \text{is a } Q_N \text{ martingale}, \quad i = 1, \dots, n.$$

(N, Q_N) is called numeraire pair.

Theorem (Pricing by martingale expectation)

X payoff of a derivative security (e.g. of a call)

Assume technical conditions (no arbitrage, payoff replicable).

Assume existence of a numeraire (N, Q_N).

The price of the payoff is given by

$$V_t(X) = N_t \, E_{Q_N}\left(\frac{X}{N_T} \,\middle|\, \mathcal{A}_t \right)$$ (discounting by the numeraire)

In particular, $$V_0(X) = N_0 \, E_{Q_N}\left(\frac{X}{N_T} \right).$$

These equations hold for <u>any</u> numeraire.

<u>Remark</u>

The pricing of American options involves stopping times.

<u>Literature:</u>

- H. Bauer: Probability Theory
- D. Lamberton, B. Lapeyre: Introduction to Stochastic Calculus Applied to Finance