Wave-Function Collapse Models: The Basic Mechanism

The time dependent Schrödinger equation is given by

$$i\hbar\tfrac{\partial}{\partial t}\psi_t \;=\; H\psi_t \tag{1}$$

with $\psi_t \in \mathcal{H}$ for all t, $\mathcal{H}$ being some Hilbert space. (1) is equivalent to

$$d\psi_t \;=\; -\tfrac{i}{\hbar}H\psi_t\,dt \tag{2}$$

Let B_t be a Brownian motion. We modify equation (2) to

$$d\psi_t \;=\; -\tfrac{i}{\hbar}H\psi_t\,dt \;+\; \lambda\big[H-\langle\psi_t,H\psi_t\rangle\big]\psi_t\,dB_t \;-\; \tfrac{\lambda^2}{2}\big[H-\langle\psi_t,H\psi_t\rangle\big]^2\psi_t\,dt \tag{3}$$

which is a nonlinear stochastic Schrödinger equation. To get a feeling for the dynamics implied by (3), we expand ψ_t with respect to the eigenfunctions of H. Thus, let $\{\varphi_n\}_{n=0}^{\infty}$ be a complete orthonormal set of eigenfunctions with

$$H\varphi_n \;=\; \varepsilon_n\,\varphi_n \tag{4}$$

and $\langle\varphi_n,\varphi_m\rangle = \delta_{n,m}$. We expand

$$\psi_t \;=\; \sum_{n=0}^{\infty} c_{n,t}\,\varphi_n \tag{5}$$

with stochastic coefficients $c_{n,t}$ and we assume that the initial state

$$\psi_0 \;=\; \sum_{n=0}^{\infty} c_{n,0}\,\varphi_n \tag{6}$$

has norm 1,

$$\|\psi_0\|^2 \;=\; \sum_{n=0}^{\infty} |c_{n,0}|^2 \;=\; 1 \tag{7}$$

We have

$$\langle\psi_t,H\psi_t\rangle \;=\; \sum_{n=0}^{\infty} \varepsilon_n\,|c_{n,t}|^2 \;=:\; \bar{\varepsilon}_t \tag{8}$$

and equation (3) becomes

$$dc_{n,t} \;=\; -i\omega_n c_{n,t}\,dt \;+\; \lambda[\varepsilon_n-\bar{\varepsilon}_t]c_{n,t}\,dB_t \;-\; \tfrac{\lambda^2}{2}[\varepsilon_n-\bar{\varepsilon}_t]^2 c_{n,t}\,dt \tag{9}$$

where we put $\omega_n := \varepsilon_n/\hbar$. Dividing (9) by $c_{n,t}$,

$$\frac{dc_{n,t}}{c_{n,t}} \;=\; \left(-i\omega_n \;-\; \tfrac{\lambda^2}{2}[\varepsilon_n - \bar{\varepsilon}_t]^2\right) dt \;+\; \lambda[\varepsilon_n - \bar{\varepsilon}_t]\, dB_t \tag{10}$$

In particular, taking the complex conjugate of (10),

$$\frac{d\bar{c}_{n,t}}{\bar{c}_{n,t}} \;=\; \left(+i\omega_n \;-\; \tfrac{\lambda^2}{2}[\varepsilon_n - \bar{\varepsilon}_t]^2\right) dt \;+\; \lambda[\varepsilon_n - \bar{\varepsilon}_t]\, dB_t \tag{11}$$

Thus, since $(dB_t)^2 = dt$,

$$\begin{aligned} d|c_{n,t}|^2 \;&=\; dc_{n,t}\,\bar{c}_{n,t} \;+\; c_{n,t}\, d\bar{c}_{n,t} \;+\; dc_{n,t}\, d\bar{c}_{n,t} \\ &=\; \left\{\tfrac{dc_{n,t}}{c_{n,t}} \;+\; \tfrac{d\bar{c}_{n,t}}{\bar{c}_{n,t}} \;+\; \tfrac{dc_{n,t}}{c_{n,t}}\,\tfrac{d\bar{c}_{n,t}}{\bar{c}_{n,t}}\right\} |c_{n,t}|^2 \\ &=\; \left\{-\,\lambda^2[\varepsilon_n - \bar{\varepsilon}_t]^2\, dt \;+\; 2\lambda[\varepsilon_n - \bar{\varepsilon}_t]\, dB_t + \lambda^2[\varepsilon_n - \bar{\varepsilon}_t]^2\, dt\right\} |c_{n,t}|^2 \\ &=\; 2\lambda[\varepsilon_n - \bar{\varepsilon}_t]\, |c_{n,t}|^2\, dB_t \end{aligned} \tag{12}$$

From (12) we get

$$\begin{aligned} d\|\psi_t\|^2 \;&=\; d\sum_{n=0}^{\infty} |c_{n,t}|^2 \\ &=\; \sum_{n=0}^{\infty} d|c_{n,t}|^2 \\ &\overset{(12)}{=}\; \sum_{n=0}^{\infty} 2\lambda[\varepsilon_n - \bar{\varepsilon}_t]\, |c_{n,t}|^2\, dB_t \\ &=\; 2\lambda\left[\bar{\varepsilon}_t - \bar{\varepsilon}_t\|\psi_t\|^2\right] dB_t \\ &=\; 2\lambda\bar{\varepsilon}_t\left[1 - \|\psi_t\|^2\right] dB_t \end{aligned} \tag{13}$$

Thus, with the initial condition $\|\psi_0\|^2 = 1$, we have $\|\psi_t\|^2 = 1$ for all $t \geq 0$. Also, as an immediate consequence of (12),

$$\mathsf{E}\left[\,|\langle\psi_t, \varphi_n\rangle|^2\,\right] \;=\; \mathsf{E}\left[\,|c_{n,t}|^2\,\right] \;=\; |c_{n,0}|^2 \tag{14}$$

which is the Born rule of quantum mechanics.

By standard Ito-calculus we have, recalling that $(dB_t)^2 = dt$,

$$\begin{aligned} d\log\left[|c_{n,t}|^2\right] \;&=\; \frac{d|c_{n,t}|^2}{|c_{n,t}|^2} \;-\; \frac{1}{2}\left\{\frac{d|c_{n,t}|^2}{|c_{n,t}|^2}\right\}^2 \\ &\overset{(12)}{=}\; 2\lambda[\varepsilon_n - \bar{\varepsilon}_t]\, dB_t \;-\; \tfrac{4\lambda^2}{2}[\varepsilon_n - \bar{\varepsilon}_t]^2\, dt \end{aligned}$$

such that we can write

$$|c_{n,t}|^2 \;=\; |c_{n,0}|^2 \times \exp\left\{2\lambda\int_0^t [\varepsilon_n - \bar{\varepsilon}_s]\, dB_s - \tfrac{4\lambda^2}{2}\int_0^t [\varepsilon_n - \bar{\varepsilon}_s]^2\, ds\right\} \tag{15}$$

Observe that (15) is not an explicit solution, since

$$\bar{\varepsilon}_t \;=\; \sum_{m=0}^{\infty} \varepsilon_m \, |c_{m,t}|^2 \tag{16}$$

depends on the coefficients $c_{m,t}$.

Perturbing with an Arbitrary Self-Adjoint Operator A:

Instead of equation (3), let us now consider, for some self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$, the equation

$$d\psi_t \;=\; -\tfrac{i}{\hbar} H\psi_t \, dt \;+\; \lambda \big[A - \langle \psi_t, A\psi_t \rangle\big] \psi_t \, dB_t \;-\; \tfrac{\lambda^2}{2} \big[A - \langle \psi_t, A\psi_t \rangle\big]^2 \psi_t \, dt \tag{17}$$

which is another nonlinear stochastic Schrödinger equation. The analysis above goes through nearly unchanged, the only thing we have to do is that we have to expand with repect to the eigenfunctions of A. Thus, let φ_n now denote a complete orthonormal set of eigenfunctions of A,

$$A\varphi_n \;=\; a_n \, \varphi_n \tag{18}$$

and $\langle \varphi_n, \varphi_m \rangle = \delta_{n,m}$. We expand

$$\psi_t \;=\; \sum_{n=0}^{\infty} c_{n,t} \, \varphi_n \tag{19}$$

with stochastic coefficients $c_{n,t}$ and we assume again that the initial state

$$\psi_0 \;=\; \sum_{n=0}^{\infty} c_{n,0} \, \varphi_n \tag{20}$$

has norm 1,

$$\|\psi_0\|^2 \;=\; \sum_{n=0}^{\infty} |c_{n,0}|^2 \;=\; 1 \tag{21}$$

We have

$$\langle \psi_t, A\psi_t \rangle \;=\; \sum_{n=0}^{\infty} a_n \, |c_{n,t}|^2 \;=:\; \bar{a}_t \tag{22}$$

and equation (17) becomes

$$dc_{n,t} \;=\; -i\omega_n c_{n,t} \, dt \;+\; \lambda[\varepsilon_n - \bar{\varepsilon}_t] c_{n,t} \, dB_t \;-\; \tfrac{\lambda^2}{2} [\varepsilon_n - \bar{\varepsilon}_t]^2 c_{n,t} \, dt \tag{23}$$

where, because of

$$\begin{aligned} H\psi_t &= \sum_{n=0}^{\infty} c_{n,t} H\varphi_n \\ &= \sum_{m,n=0}^{\infty} c_{n,t}\langle H\varphi_n, \varphi_m\rangle\, \varphi_m \\ &= \sum_{m,n=0}^{\infty} c_{m,t}\langle H\varphi_m, \varphi_n\rangle\, \varphi_n \\ &= \sum_{m,n=0}^{\infty} \frac{\langle Hc_{m,t}\varphi_m, c_{n,t}\varphi_n\rangle}{|c_{n,t}|^2}\, c_{n,t}\varphi_n \end{aligned} \tag{24}$$

we now have to define

$$\omega_{n,t} \;:=\; \tfrac{1}{\hbar}\sum_{m=0}^{\infty} \frac{\langle Hc_{m,t}\varphi_m, c_{n,t}\varphi_n\rangle}{|c_{n,t}|^2} \tag{25}$$

which is no longer necessarily a real quantity. Dividing (23) by $c_{n,t}$ again, we arrive at

$$\tfrac{dc_{n,t}}{c_{n,t}} \;=\; \left(-i\omega_{n,t} \;-\; \tfrac{\lambda^2}{2}[a_n - \bar{a}_t]^2\right) dt \;+\; \lambda[a_n - \bar{a}_t]\, dB_t \tag{26}$$

In particular, taking the complex conjugate of (26),

$$\tfrac{d\bar{c}_{n,t}}{\bar{c}_{n,t}} \;=\; \left(+i\bar{\omega}_{n,t} \;-\; \tfrac{\lambda^2}{2}[a_n - \bar{a}_t]^2\right) dt \;+\; \lambda[a_n - \bar{a}_t]\, dB_t \tag{27}$$

such that

$$\begin{aligned} d|c_{n,t}|^2 &= dc_{n,t}\,\bar{c}_{n,t} \;+\; c_{n,t}\,d\bar{c}_{n,t} \;+\; dc_{n,t}\,d\bar{c}_{n,t} \\ &= \left\{\tfrac{dc_{n,t}}{c_{n,t}} \;+\; \tfrac{d\bar{c}_{n,t}}{\bar{c}_{n,t}} \;+\; \tfrac{dc_{n,t}}{c_{n,t}}\,\tfrac{d\bar{c}_{n,t}}{\bar{c}_{n,t}}\right\} |c_{n,t}|^2 \\ &= \left\{2\,\mathrm{Im}(\omega_{n,t})\,dt \;-\; \lambda^2[a_n - \bar{a}_t]^2\,dt \;+\; 2\lambda[a_n - \bar{a}_t]\,dB_t + \;\lambda^2[a_n - \bar{a}_t]^2\,dt\right\} |c_{n,t}|^2 \\ &= \left\{2\,\mathrm{Im}(\omega_{n,t})\,dt \;+\; 2\lambda[a_n - \bar{a}_t]\,dB_t\right\} |c_{n,t}|^2 \end{aligned} \tag{28}$$

There are 2 cases: If A commutes with H, $[A, H] = AH - HA = 0$, then there is a simultaneous set of eigenfunctions and ω_n is again a real quantity. In that case the situation is identical to the case discussed above. This corresponds to the case that the physical observables A and H can be both measured precisely. If A does not commute with H, there is an uncertainty relation which states that A and H cannot be simultaneously measured to arbitrary precision. In that case, the imaginary part in (28) is really there and impacts the dynamics. Observe however that in any case

$$\sum_{n=0}^{\infty} \omega_{n,t}\,|c_{n,t}|^2 \;=\; \tfrac{1}{\hbar}\sum_{m,n=0}^{\infty} \langle Hc_{m,t}\varphi_m, c_{n,t}\varphi_n\rangle \;\in\; \mathbb{R} \tag{29}$$

is a real quantity such that, as above,

$$
\begin{aligned}
d\|\psi_t\|^2 &= d\sum_{n=0}^{\infty}|c_{n,t}|^2 \\
&= \sum_{n=0}^{\infty} d|c_{n,t}|^2 \\
&\overset{(28)}{=} \sum_{n=0}^{\infty}\Big\{2\,\mathrm{Im}(\omega_{n,t})\,dt \;+\; 2\lambda[a_n-\bar{a}_t]\,dB_t\Big\}\,|c_{n,t}|^2 \\
&= 2\mathrm{Im}\left[\sum_{n=0}^{\infty}\omega_{n,t}\,|c_{n,t}|^2\right]dt \;+\; \sum_{n=0}^{\infty}2\lambda[a_n-\bar{a}_t]\,|c_{n,t}|^2\,dB_t \\
&\overset{(29)}{=} \sum_{n=0}^{\infty}2\lambda[a_n-\bar{a}_t]\,|c_{n,t}|^2\,dB_t \\
&= 2\lambda\big[\bar{a}_t-\bar{a}_t\|\psi_t\|^2\big]\,dB_t \\
&= 2\lambda\bar{a}_t\big[1-\|\psi_t\|^2\big]\,dB_t \qquad (30)
\end{aligned}
$$

Thus, again, with the initial condition $\|\psi_0\|^2=1$, we have $\|\psi_t\|^2=1$ for all $t\geq 0$.