Functional Analysis for Differential Equations

Lectures in the Summer School 2016

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September 23, 2016

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1 Introduction

Differential equations play a key role in applied mathematics and, they are very challenging – for two reasons:
From the practical point of view, they occur in many models of systems in physics, chemistry, quantitative finance and the engineering sciences. Explicit solution formulas, however, are hardly available. Standard methods like variation of constants, separation of variables, GREEN functions or integral transformations are usually restricted to very special cases in regard to the geometry or the equations.

Functional analysis offers a way out for some partial differential equations. The key idea is based on re-formulating the "classical" differential equation as an integral equation.
DIRICHLET (1805 – 1859) opened a completely new door in the theory of partial differential equations (PDEs). He investigated minimizers of real-valued functionals (usually interpreted as an "energy functional") and characterized them as solutions to related PDEs.

In these lectures, we are going to follow this track in the special case of one independent variable, i.e., our considerations are restricted to *ordinary* differential equations (for quite a while, at least). We will find a new criterion on solutions called *weak solutions*.
They satisfy the same general integral condition as "classical" solutions do, but they might not be sufficiently differentiable in the classical sense. From the analytical point of view, their essential advantage over classical solutions is that their general existence can be proved in a quite convenient way – if we are ready to accept the (slightly more abstract) setting of HILBERT spaces.
The final "surprise" is that this abstract approach also lays the foundations for standard numerical methods. Indeed, we present the analytic basis of finite element methods. All these considerations can be applied to functions of more than one independent variable, i.e., to certain *partial* differential equations.

2 Boundary Value Problems for Ordinary Diff. Equations

In the field of ordinary differential equations, standard theorems usually concern *initial* value problems. The theorem of CAUCHY–LIPSCHITZ (a.k.a. PICARD-LINDELÖF), for example, guarantees both existence and uniqueness of the solution $u : J \longrightarrow \mathbb{R}^n$ of

$$\wedge \begin{cases} u'(t) &= f(t, u(t)) \qquad \text{for all } t \in J \\ u(t_0) &= u_0 \in \mathbb{R}^n \end{cases} \tag{$*$}$$

whenever $J \subset \mathbb{R}$ is a nonempty interval, $t_0 \in J$ and the continuous function $f : J \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is LIPSCHITZ-continuous with respect to its second (i.e., vector-valued) argument.

Its proof is known to be very constructive. Indeed, the EULER method leads to an approximating sequence of continuous and piecewise linear functions $J \longrightarrow \mathbb{R}^n$ which proves to be a CAUCHY sequence with respect to the supremum norm. Hence there is a limit function $u \in C^0(J, \mathbb{R}^n)$ and, it satisfies the integral equation

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s)) \; ds \qquad \text{for every } t \in J.$$

The continuity of both $u(\cdot)$ and $f(\cdot, \cdot)$ implies that the composition

$$J \longrightarrow \mathbb{R}^n, \qquad s \longmapsto f(s, u(s))$$

is also continuous. Now we conclude from the fundamental theorem of calculus that

$$J \longrightarrow \mathbb{R}^n, \qquad t \longmapsto \int_{t_0}^{t} f(s, u(s)) \; ds$$

is differentiable (in the well-known sense) and so is the limit $u : J \longrightarrow \mathbb{R}^n$. Together with the observation $u(t_0) = u_0$, it completes the proof that $u(\cdot)$ solves the initial value problem $(*)$.

This approximating method is very difficult to adapt to *boundary* value problems (maybe even impossible). Hence we need another approach.

2.I A Simple Boundary Value Problem

The following boundary value problem

$$\wedge \begin{cases} u''(t) &= f(t) \quad \text{for every } t \in]0,1[\\ u(0) &= 0 \\ u(1) &= 0 \end{cases}$$

serves us as a representative example of a scalar second-order differential equation with DIRICHLET boundary conditions. The wanted function $u : [0,1] \longrightarrow \mathbb{R}$ describes the position of a mass point on a horizontal line, for example, whose acceleration is prescribed as a function $f(\cdot)$ of time. Furthermore the mass point is at the origin position, i.e., $u(t) = 0$, both at the initial time $t = 0$ and the end time $t = 1$.

The theorem of CAUCHY-LIPSCHITZ cannot be applied directly because the initial velocity $u'(0)$ is not given. For every parameter $u_1 \in \mathbb{R}$, there exists a unique solution $v : [0,1] \longrightarrow \mathbb{R}$ of

$$\wedge \begin{cases} v''(t) &= f(t) \quad \text{for every } t \in]0,1[\\ v(0) &= 0 \\ v'(0) &= u_1 \end{cases} \tag{$*$}$$

and, this initial-value problem of second order is so easy that we can even specify it explicitly due to the fundamental theorem of calculus

$$v'(t) = u_1 + \int_0^t f(s)\ ds \qquad \text{for all } t \in [0,1]$$

$$\Longrightarrow \quad v(t) = \int_0^t v'(s)\ ds = u_1 \cdot t + \int_0^t \int_0^s f(r)\ dr\ ds \quad \text{for all } t \in [0,1].$$

Finally we choose the parameter $u_1 \in \mathbb{R}$ so that $v(\cdot)$ also satisfies the remaining boundary condition, i.e., $v(1) = 0$:

$$u_1 = -\int_0^1 \int_0^s f(r)\ dr\ .$$

Hence, $u : [0,1] \longrightarrow \mathbb{R},\ t \longmapsto -t \cdot \int_0^1 \int_0^s f(r)\ dr + \int_0^t \int_0^s f(r)\ dr\ ds$ is the wanted solution of $(*)$.

Admittedly this example $(*)$ is simple. But it helps us to realize a first approach to boundary value problems of ordinary differential equations: Reformulate it as an initial value problem with additional parameters (such as u_1 here), solve the latter by means of standard ODE tools and then choose the parameters so that the missing boundary condition are also satisfied (if possible). This is the gist of so-called *shooting methods*.

This approach, however, is difficult to extend to *partial* differential equations since they consider functions of more than one variable. Hence we now study a completely different method for the same toy example.

2.II DIRICHLET's principle

Many effects in nature can be described in terms of minimizing some scalar energy. PIERRE DE FERMAT (1607– 1665) was one of the first researchers who expressed the final path of light in terms of minimal time. Several other famous scientists like LEONHARD EULER (1707 – 1783), JOSEPH-LOUIS LAGRANGE (1736 – 1813) and Sir WILLIAM ROWAN HAMILTON (1805 – 1865) followed and suggested similar statements of minimizing various functionals in mechanical systems.

PETER GUSTAV LEJEUNE DIRICHLET (1805 – 1859) is said to initiate the notion that solutions of some partial differential equations (so-called *harmonic functions*) can be characterized as minimizers of an appropriate functional. We now discuss this principle in the special case of one real variable:

Definition 2.1 *For any nonempty bounded closed subset $M \subset \mathbb{R}$, define*

$$C_0^0(M) \;=\; \left\{ v : M \longrightarrow \mathbb{R} \;\middle|\; v(\cdot) \text{ is continuous and } v(x) = 0 \text{ for all } x \in \partial M \right\}.$$

Proposition 2.2 (DIRICHLET's principle in one variable)
For any function $f \in C^0([0,1])$ given, suppose that

$$\begin{aligned} \Psi : \quad C_0^0([0,1]) \cap C^1(]0,1[) \;&\longrightarrow\; \mathbb{R}, \\ v(\cdot) \;&\longmapsto\; \tfrac{1}{2} \cdot \int_0^1 v'(t)^2 \; dt \;+\; \int_0^1 f(t) \cdot v(t) \; dt \end{aligned}$$

has its global minimum at $u(\cdot)$. Furthermore assume $u \in C^2([0,1])$.

Then this minimizer $u : [0,1] \longrightarrow \mathbb{R}$ is a solution to the boundary value problem

$$\wedge \begin{cases} u''(t) &= f(t) \quad \text{for every } t \in]0,1[\\ u(0) &= 0 \\ u(1) &= 0. \end{cases}$$

Proof: The basic idea is to adapt the concept of directional derivative. Fix an auxiliary function $\varphi \in C_0^0([0,1]) \cap C^1(]0,1[)$ arbitrarily. For each $h \in \mathbb{R}$, the function

$$v_h \;:=\; u(\cdot) + h \cdot \varphi(\cdot) : \quad [0,1] \longrightarrow \mathbb{R}, \qquad t \longmapsto u(t) + h \cdot \varphi(t)$$

belongs to $C_0^0([0,1]) \cap C^1(]0,1[)$, i.e., to the domain of Ψ. By assumption, Ψ has its global minimum at $u(\cdot)$ and so, the composition

$$\mathbb{R} \longrightarrow \mathbb{R}, \qquad h \longmapsto \Psi(v_h) \overset{\text{Def.}}{=} \Psi(u + h \cdot \varphi)$$

has its global minimum at $h = 0$:

$$\tfrac{1}{2} \int_0^1 v_h'(t)^2 \, dt \;+\; \int_0^1 f(t) \cdot v_h(t) \, dt \;\geq\; \tfrac{1}{2} \int_0^1 u'(t)^2 \, dt \;+\; \int_0^1 f(t) \cdot u(t) \, dt$$

for all $h \in \mathbb{R}$. We conclude for every $h \in \mathbb{R}$

$$\begin{aligned}
& \int_0^1 \Big(\tfrac{1}{2}\, v_h'(t)^2 - \tfrac{1}{2}\, u'(t)^2 \;+\; f(t) \cdot \big(v_h(t) - u(t)\big)\Big) \, dt \;\geq\; 0 \\
\Longleftrightarrow \quad & \int_0^1 \Big(\tfrac{1}{2}\, \big(u'(t) + h \cdot \varphi'(t)\big)^2 - \tfrac{1}{2}\, u'(t)^2 \;+\; f(t) \cdot \; h \cdot \varphi(t) \Big) \, dt \;\geq\; 0 \\
\Longleftrightarrow \quad & \int_0^1 \Big(u'(t) \cdot h \cdot \varphi'(t) \;+\; \tfrac{1}{2}\, h^2 \cdot \varphi'(t)^2 \;+\; f(t) \cdot \; h \cdot \varphi(t) \Big) \, dt \;\geq\; 0 \\
\Longleftrightarrow \quad & \tfrac{1}{2}\, h^2 \cdot \int_0^1 \varphi'(t)^2 \; dt \;+\; h \cdot \int_0^1 \Big(u'(t) \cdot \varphi'(t) \;+\; f(t) \cdot \varphi(t) \Big) \, dt \;\geq\; 0.
\end{aligned}$$

Restricting our considerations to any $h > 0$, in particular, we obtain

$$\tfrac{1}{2}\, h \cdot \int_0^1 \varphi'(t)^2 \; dt \;+\; \int_0^1 \Big(u'(t) \cdot \varphi'(t) \;+\; f(t) \cdot \varphi(t) \Big) \; dt \;\;\geq\;\; 0.$$

The second integral on the left-hand side can be simplified since $\varphi(0) = 0 = \varphi(1)$ and $u(\cdot) \in C^2([0,1])$. Indeed, the integration by parts leads to

$$\begin{aligned}
\int_0^1 u'(t) \cdot \varphi'(t) \; dt \;&=\; \Big[u'(t) \cdot \varphi(t) \Big]_{t=0}^{t=1} - \int_0^1 u''(t) \cdot \varphi(t) \; dt \\
&=\; \qquad 0 \qquad - \int_0^1 u''(t) \cdot \varphi(t) \; dt
\end{aligned}$$

and so,

$$\tfrac{1}{2}\, h \cdot \int_0^1 \varphi'(t)^2 \; dt \;+\; \int_0^1 \Big(- \, u''(t) \;+\; f(t) \Big) \cdot \varphi(t) \; dt \;\;\geq\;\; 0$$

for all $h > 0$ is a necessary condition on the minimizer $u \in C_0^0([0,1]) \cap C^2$ of Ψ.
The limit for $h \longrightarrow 0^+$ reveals

$$\int_0^1 \Big(- u''(t) + f(t) \Big) \cdot \varphi(t) \; dt \;\; \geq \;\; 0.$$

It is worth mentioning here that the first factor $-u''(t) + f(t)$ is a continuous function $[0,1] \longrightarrow \mathbb{R}$ whereas the auxiliary function $\varphi \in C_0^0([0,1]) \cap C^1(]0,1[)$ had been fixed arbitrarily. According to an indirect consequence of continuity, this inequality can hold for all $\varphi \in C_0^0([0,1]) \cap C^1(]0,1[)$ only if

$$-u''(t) + f(t) \;=\; 0 \qquad \text{for every } t \in [0,1]$$

is satisfied, i.e., $u(\cdot)$ fulfils the claimed differential equation.
Finally the boundary conditions $u(0) = 0 = u(1)$ are assumed and so, the global minimizer $u \in C_0^0([0,1]) \cap C^1(]0,1[)$ of $\Psi(\cdot)$ with $u \in C^2([0,1])$ solves the full boundary value problem.

□

This approach, however, implies several questions:

- $\Psi : C_0^0([0,1]) \cap C^1(]0,1[) \longrightarrow \mathbb{R}$ proves to be bounded from below. Is its infimum really attained ? In other words: Does Ψ really have a minimizer ?

- Suppose that $u \in C_0^0([0,1]) \cap C^1(]0,1[)$ minimizes the functional $\Psi(\cdot)$. Does this $u(\cdot)$ also fulfil the stronger regularity condition $u \in C^2([0,1])$? (We need this additional regularity of $u(\cdot)$ for the integration by parts in the proof.)

The answers to these questions are really nontrivial and so, in these lectures we completely dispense with their analytical details.

We are going to focus on the very first step instead: How to characterize (and verify) a candidate for the wanted solution ... The basic idea is to formulate a generalised problem satisfying two criteria:

- Its characterizing conditions are weaker than in the original boundary value problem so that the existence of solution can be proved in general.
- Whenever a solution of the generalised problem is found, it is also a candidate for the original boundary value problem.
  In more detail, the generalised problem is based on necessary conditions, but there might be additional properties (like supplementary differentiability) to verify for concluding that a solution to the generalised problem also solves the original boundary value problem.

The focus of this course is on the first aspect, not the second one.

2.III Weak derivatives

We continue with the simple example of a boundary value problem

$$\wedge \begin{cases} u''(t) &= f(t) \quad \text{for every } t \in]0,1[\\ u(0) &= 0 \\ u(1) &= 0. \end{cases}$$

The first step on our way to a "generalised" problem is to reduce the regularity of functions under consideration. In our toy example, every "classical" solutions $u : [0,1] \longrightarrow \mathbb{R}$ should be twice differentiable in $]0,1[$ and avoid jumps at the boundary of $[0,1]$ so that the boundary condition $u(0) = 0 = u(1)$ gives us useful qualitative information about $u(\cdot)$ close to the boundary. Both aspects lead to the standard choice for $u(\cdot)$

$$u \in C_0^0([0,1]) \cap C^2(]0,1[) .$$

A closer look at the proof of DIRICHLET's principle (Proposition 2.2 on page 5), however, reveals two perspectives how to reduce the regularity of considered functions:

(i) We prefer integrals to derivatives.
Indeed, whenever proving the CAUCHY-LIPSCHITZ theorem for ODE initial value problems, the starting point is usually to formulate the initial value problem as an integral equation, i.e., for any functions $x \in C^0([0,1],\mathbb{R}^n)$ and $g \in C^0([0,1]\times\mathbb{R}^n,\mathbb{R}^n)$,

$$\wedge \begin{cases} x' &= g(t,x) \quad \text{in }]0,1[\\ x(0) &= x_0 \end{cases}$$

$$\Longleftrightarrow \quad \forall\, t \in [0,1] : \quad x(t) = x_0 + \int_0^t g\big(s,\ x(s)\big)\ ds.$$

This equivalence results from the fundamental theorem of calculus due to the continuity of the composition $[0,1] \longrightarrow \mathbb{R}^n$, $s \longmapsto g\big(s,\ x(s)\big)$.
Considering just the integral equation, however, we do not require the continuity of $g(\cdot,x)$ or the differentiability of $x(\cdot)$. This condition already makes sense for integrable functions.

In regard to our second-order boundary value problem, the proof of DIRICHLET's principle considers derivatives of both the minimizer $u(\cdot)$ and the auxiliary function $\varphi(\cdot)$ just in integrals, not separately – until the very end.

(ii) Integration by parts can help us reducing the order.
In the proof of DIRICHLET's principle, integration by parts provides the relevant connection between the second-order differential equation and the first-order derivative in the minimized functional $\Psi(\cdot)$ ... and this step is used in the very end only.

Until then, the minimizer $u(\cdot)$ is characterized in terms of the condition

$$\tfrac{1}{2}\, h \cdot \int_0^1 \varphi'(t)^2 \; dt \;+\; \int_0^1 \Big(u'(t) \cdot \varphi'(t) \;+\; f(t) \cdot \varphi(t) \Big) \; dt \;\;\geq\;\; 0$$

for every parameter $h > 0$ and any auxiliary function $\varphi \in C^0_0([0,1]) \cap C^1(]0,1[)$. The limit for $h \longrightarrow 0^+$ leads to

$$\int_0^1 \Big(u'(t) \cdot \varphi'(t) \;+\; f(t) \cdot \varphi(t) \Big) \; dt \;\;\geq\;\; 0$$

for every $\varphi \in C^0_0([0,1]) \cap C^1(]0,1[)$. If we consider both arbitrary $\varphi(\cdot)$ and its negative $-\varphi(\cdot)$, this condition implies equality, i.e., $u(\cdot)$ satisfies

$$\int_0^1 \Big(u'(t) \cdot \varphi'(t) \;+\; f(t) \cdot \varphi(t) \Big) \; dt \;\;=\;\; 0$$

for every $\varphi \in C^0_0([0,1]) \cap C^1(]0,1[)$.
This is a so-called *variational equation* for $u(\cdot)$ as it holds for each function $\varphi(\cdot)$ in a specified (possibly very large) class. It is worth mentioning that the considered derivatives are of order ≤ 1 (not 2 as in the original boundary value problem).

These two observations motivate us to introduce the following definitions:

Definition 2.3 *Let $J \subset \mathbb{R}$ be a nonempty interval. A function $x : J \longrightarrow \mathbb{R}$ is called* absolutely continuous *if there exists a* LEBESGUE *integrable function $g : J \longrightarrow \mathbb{R}$ such that*

$$x(t_1) - x(t_0) = \int_{[t_0, t_1]} g(s) \; d\mathcal{L}^1 s$$

holds for any real $t_0, t_1 \in J$ with $t_0 \leq t_1$. Every LEBESGUE *integrable function $g : J \longrightarrow \mathbb{R}$ satisfying this representation of $x(\cdot)$ is called its* weak derivative*. $AC(J, \mathbb{R})$ denotes the set of all absolutely continuous functions $J \longrightarrow \mathbb{R}$.*

Remark 2.4 (1.) Every absolutely continuous function $x : J \longrightarrow \mathbb{R}$ is continuous due to the general properties of LEBESGUE integrals.

(2.) The fundamental theorem of calculus implies for any $x \in AC(J, \mathbb{R})$ and its weak derivative $g : J \longrightarrow \mathbb{R}$ (on an interval $J \subset \mathbb{R}$ with more than one point): If $g(\cdot)$ is continuous in addition then $x(\cdot)$ is differentiable in the well established sense that

$$x'(t_0) := \lim_{\substack{t \to t_0 \\ (t \neq t_0)}} \frac{x(t) - x(t_0)}{t - t_0}$$

exists and is equal to $g(t)$ for every $t \in J$.

Proposition 2.5 *Let $J = [a, b] \subset \mathbb{R}$ be an interval and suppose $g : J \longrightarrow \mathbb{R}$ to be* LEBESGUE *integrable. Then the following statements about $x : J \longrightarrow \mathbb{R}$ are equivalent:*

(1.) *$x(\cdot)$ is absolutely continuous with the weak derivative $g(\cdot)$ (Definition 2.3).*

(2.) *$x(\cdot)$ is continuous and for any auxiliary function $\varphi \in C^1(J, \mathbb{R})$, it holds*

$$\int_J \left(x(s) \cdot \varphi'(s) + g(s) \cdot \varphi(s) \right) \; d\mathcal{L}^1 s = x(b) \cdot \varphi(b) - x(a) \cdot \varphi(a).$$

(3.) *$x(\cdot)$ is continuous and for any $\psi \in C^1(J, \mathbb{R}) \cap C^0_0(J)$,*

$$\int_J x(s) \cdot \psi'(s) \; d\mathcal{L}^1 s = - \int_J g(s) \cdot \psi(s) \; d\mathcal{L}^1 s.$$

Proof: "(1.) $\Longrightarrow$ (2.)" For every function $\varphi \in C^1(J, \mathbb{R})$ and arbitrary $c, d \in J$ $(c < d)$, the fundamental theorem of calculus implies

$$x(d) \cdot \varphi(d) - x(c) \cdot \varphi(c) = x(d) \cdot \Big(\varphi(d) - \varphi(c)\Big) + \Big(x(d) - x(c)\Big) \cdot \varphi(c)$$
$$= x(d) \cdot \int_c^d \varphi'(s)\, ds + \int_{[c,d]} g(s)\, d\mathcal{L}^1 s \cdot \varphi(c).$$

The continuous functions $x(\cdot)$ and $\varphi(\cdot)$ are even uniformly continuous in the compact interval $J = [a, b]$. Hence for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $s, t \in J$:

$$|s - t| < \delta \quad \Longrightarrow \quad |x(t) - x(s)| \le \varepsilon \;\wedge\; |\varphi(t) - \varphi(s)| \le \varepsilon.$$

Now we obtain for all $c, d \in J$ with $0 < d - c < \delta$

$$\Big| x(d) \cdot \varphi(d) - x(c) \cdot \varphi(c) - \int_{[c,d]} \big(x(s) \cdot \varphi'(s) + g(s) \cdot \varphi(s)\big)\; d\mathcal{L}^1 s \Big|$$
$$\le \int_{[c,d]} |x(d) - x(s)| \cdot |\varphi'(s)|\; d\mathcal{L}^1 s + \int_{[c,d]} |g(s)| \cdot |\varphi(s) - \varphi(c)|\; d\mathcal{L}^1 s$$
$$\le \varepsilon \cdot \int_{[c,d]} |\varphi'(s)|\; d\mathcal{L}^1 s + \varepsilon \cdot \int_{[c,d]} |g(s)|\; d\mathcal{L}^1 s.$$

Choose an arbitrary partition of the interval $J = [a, b]$

$$a = t_0 < t_1 < t_2 < \ldots < t_n = b \quad \text{with} \quad \max_j \big|t_j - t_{j-1}\big| < \delta.$$

Then the standard argument of a telescoping sum leads to

$$\Big| x(b) \cdot \varphi(b) - x(a) \cdot \varphi(a) - \int_J \big(x(s) \cdot \varphi'(s) + g(s) \cdot \varphi(s)\big)\; d\mathcal{L}^1 s \Big|$$
$$\le \sum_{j=1}^n \Big| x(t_j) \cdot \varphi(t_j) - x(t_{j-1}) \cdot \varphi(t_{j-1}) - \int_{[t_{j-1}, t_j]} (x \cdot \varphi' + g \cdot \varphi)\; d\mathcal{L}^1 s \Big|$$
$$\le \sum_{j=1}^n \Big(\varepsilon \cdot \int_{[t_{j-1}, t_j]} |\varphi'(s)|\; d\mathcal{L}^1 s + \varepsilon \cdot \int_{[t_{j-1}, t_j]} |g(s)|\; d\mathcal{L}^1 s\Big)$$
$$\le \varepsilon \cdot \int_a^b |\varphi'(s)|\; ds + \varepsilon \cdot \int_J |g(s)|\; d\mathcal{L}^1 s$$

with $\varepsilon > 0$ fixed arbitrarily small.

"(2.) $\Longrightarrow$ (3.)" Obviously, statement (3.) is a special case of statement (2.).

"(3.) $\Longrightarrow$ (1.)" Choose $t_0, t_1 \in J$ ($t_0 < t_1$) and $\varepsilon \in \left]0, \frac{t_1 - t_0}{2}\right[$ arbitrarily. Due to the continuity of $x : J \longrightarrow \mathbb{R}$ at t_0 and t_1, there exists some $\delta \in \,]0, \varepsilon[$ such that for all $s \in J$,

$$\begin{cases} |s - t_0| \;\le\; \delta & \Longrightarrow \quad |x(s) - x(t_0)| \;\le\; \varepsilon, \\ |s - t_1| \;\le\; \delta & \Longrightarrow \quad |x(s) - x(t_1)| \;\le\; \varepsilon. \end{cases}$$

The auxiliary function $\psi : J \longrightarrow [0, 1]$ is defined in the following piecewise way:

$$\psi(s) \;:=\; \begin{cases} 0 & \text{for} \quad a \le s < t_0 \\ \exp\left(- \frac{1}{\left(\frac{s - t_0}{s - t_0 - \delta}\right)^4} \right) & \text{for} \quad t_0 < s < t_0 + \delta \\ 1 & \text{for} \quad t_0 + \delta \le s \le t_1 - \delta \\ \exp\left(- \frac{1}{\left(\frac{s - t_1}{s - t_1 + \delta}\right)^4} \right) & \text{for} \quad t_1 - \delta < s < t_1 \\ 0 & \text{for} \quad t_1 \le s \le b. \end{cases}$$

$\psi(\cdot)$ is increasing in $[t_0, t_0 + \delta]$ and decreasing in $[t_1 - \delta, t_1]$. Moreover, $\psi(\cdot)$ is smooth in J and so, the assumption (3.) leads to

$$\int_J x(s) \cdot \psi'(s) \; d\mathcal{L}^1 s \;\;=\;\; - \int_J g(s) \cdot \psi(s) \; d\mathcal{L}^1 s.$$

Due to $\psi(\cdot) = 1$ in $[t_0 + \delta, t_1 - \delta]$ and $\psi(\cdot) = 0$ in $[a, t_0] \cup [t_1, b]$, we obtain

$$\begin{aligned} &\int_{[t_0,\, t_0+\delta]} x(s) \cdot \psi'(s) \; d\mathcal{L}^1 s \;+\; \int_{[t_1-\delta,\, t_1]} x(s) \cdot \psi'(s) \; d\mathcal{L}^1 s \\ &\;=\; - \int_{[t_0,\, t_0+\delta]} g \cdot \psi \; d\mathcal{L}^1 s \;-\; \int_{[t_0+\delta,\, t_1-\delta]} g \; d\mathcal{L}^1 s \;-\; \int_{[t_1-\delta,\, t_1]} g \cdot \psi \; d\mathcal{L}^1 s. \end{aligned}$$

In regard to the LEBESGUE integrals on the left-hand side, we conclude from the criterion of δ

$$\begin{aligned} \Big| \int_{[t_0,\, t_0+\delta]} x(s) \cdot \psi'(s) \; d\mathcal{L}^1 s \;-\; x(t_0) \Big| \;&\le\; \int_{[t_0,\, t_0+\delta]} \varepsilon \; |\psi'(s)| \; d\mathcal{L}^1 s \;\;=\;\; \varepsilon, \\ \Big| \int_{[t_1-\delta,\, t_1]} x(s) \cdot \psi'(s) \; d\mathcal{L}^1 s \;-\; x(t_1) \Big| \;&\le\; \int_{[t_1-\delta,\, t_1]} \varepsilon \; |\psi'(s)| \; d\mathcal{L}^1 s \;\;=\;\; \varepsilon. \end{aligned}$$

In regard to the first and the third integral on the right-hand side, the inequality $0 \le \psi(\cdot) \le 1$ implies

$$-\int_{[t_0,\,t_0+\delta]} |g| \ d\mathcal{L}^1 s \ \le \ \int_{[t_0,\,t_0+\delta]} g \cdot \psi \ d\mathcal{L}^1 s \ \le \ \int_{[t_0,\,t_0+\delta]} |g| \ d\mathcal{L}^1 s,$$
$$-\int_{[t_1-\delta,\,t_1]} |g| \ d\mathcal{L}^1 s \ \le \ \int_{[t_1-\delta,\,t_1]} g \cdot \psi \ d\mathcal{L}^1 s \ \le \ \int_{[t_1-\delta,\,t_1]} |g| \ d\mathcal{L}^1 s.$$

Due to $0 < \delta < \varepsilon < \frac{t_1 - t_0}{2}$, we obtain

$$\Big| x(t_1) \ - \ x(t_0) \ - \ \int_{[t_0,\,t_1]} g(s) \ d\mathcal{L}^1 s \Big|$$
$$\le \ 2\,\varepsilon \ + \ 2 \cdot \int_{[t_0,\,t_0+\varepsilon]} |g(s)| \ d\mathcal{L}^1 s \ + \ 2 \cdot \int_{[t_1-\varepsilon,\,t_1]} |g(s)| \ d\mathcal{L}^1 s.$$

The parameter $\varepsilon \in \ \Big]0, \ \frac{t_1 - t_0}{2}\Big[$ had been chosen arbitrarily small and so,

$$\Big| x(t_1) \ - \ x(t_0) \ - \ \int_{[t_0,\,t_1]} g(s) \ d\mathcal{L}^1 s \Big| \ = \ 0$$

holds for any $t_0, t_1 \in J$ with $t_0 < t_1$. □

2.IV The "weak" formulation of $u'' = f$

In § 2.III, we have already mentioned the preliminary goal of an auxiliary problem which generalises the boundary value problem

$$\wedge \begin{cases} u''(t) &= f(t) \quad \text{for every } t \in]0,1[\\ u(0) &= 0 \\ u(1) &= 0 \end{cases}$$

and whose solution can be found by means of functional analysis. Furthermore, observation (ii) (page 10) underlines the special role of the variational equation

$$\int_0^1 \Big(u'(t) \cdot \varphi'(t) \; + \; f(t) \cdot \varphi(t) \Big) \; dt \;\; = \;\; 0 \qquad (*)$$

for every $\varphi \in C_0^0([0,1]) \cap C^1(]0,1[)$. We now use it as a starting point for the generalised problem and combine it with the concept of weak derivatives of first order introduced in Definition 2.3 (on page 11).

The first (formal) step consists in reducing the regularity of the functions, i.e., from now on, we consider $u, \; \varphi \in AC([0,1])$ instead of $u, \; \varphi \in C^1(]0,1[)$. Together with the boundary conditions

$$\wedge \begin{cases} u(0) = 0 = u(1), \\ \varphi(0) = 0 = \varphi(1), \end{cases}$$

the intersection $AC([0,1]) \cap C_0^0([0,1])$ suggests itself as the new basic set for the functions u, φ. It is a superset of the original choice $C_0^0([0,1]) \cap C^1(]0,1[)$ due to Remark 2.4 (on page 11) and so, it should also contain "classical" solutions to the original boundary value problem.

The second (formal) step focuses on the variational equation $(*)$ and its extension to $u, \varphi \in AC([0,1]) \cap C_0^0([0,1])$. In particular, integrals are now understood in the LEBESGUE sense (if required) and, the first part

$$\int_{[0,1]} u'(t) \cdot \varphi'(t) \;\; d\mathcal{L}^1 t$$

refers to the *weak* derivatives of $u(\cdot)$ and $\varphi(\cdot)$ respectively.

There is an measure-theoretic obstacle though: If both $u' : [0,1] \longrightarrow \mathbb{R}$ and $\varphi' : [0,1] \longrightarrow \mathbb{R}$ are LEBESGUE integrable, then their pointwise product does not share this property in general. There are simple counterexamples. Hence we have to modify our choice of the basic function set.

Whenever the LEBESGUE integrability of products causes difficulties, HÖLDER's inequality proves to be useful for suggesting appropriate restrictions:

Lemma 2.6 (HÖLDER's inequality)

Fix any $p, q \in]1, \infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$. Suppose for the LEBESGUE measurable functions $v, w : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ that both $|v|^p$ and $|w|^q$ are LEBESGUE integrable.

Then the pointwise product $v \cdot w : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ is LEBESGUE integrable and

$$\Big| \int_{\mathbb{R}} v(s) \cdot w(s) \; d\mathcal{L}^1 s \Big| \;\leq\; \Big(\int_{\mathbb{R}} |v|^p \; d\mathcal{L}^1 s \Big)^{\frac{1}{p}} \; \Big(\int_{\mathbb{R}} |w|^q \; d\mathcal{L}^1 s \Big)^{\frac{1}{q}} \;\stackrel{\text{Def.}}{=}\; \|v\|_{L^p(\mathbb{R})} \; \|w\|_{L^q(\mathbb{R})} \, .$$

The additional condition $p = q$ leads to $p = q = 2$ as the only common choice for $p, q \in]1, \infty[$ satisfying the assumptions of HÖLDER's inequality. This observation arouses our interest in square (LEBESGUE) integrable functions:

Definition 2.7 *Let $J \subset \mathbb{R}$ be a nonempty interval. A LEBESGUE measurable function $v : J \longrightarrow \overline{\mathbb{R}}$ is called* square integrable *or* of bounded second moment *if the pointwise square $v^2 : J \longrightarrow \mathbb{R} \cup \{\infty\}$ is LEBESGUE integrable. Set*

$$\|v\|_{L^2(J)} \;:=\; \sqrt{\int_J |v(s)|^2 \; d\mathcal{L}^1 s} \; .$$

$L^2(J)$ consists of all square integrable functions $J \longrightarrow \overline{\mathbb{R}}$ (with any two functions v, w of them being identified whenever $\|v - w\|_{L^2(J)} = 0$).

Remark 2.8 If the nonempty interval $J \subset \mathbb{R}$ is bounded in addition, then HÖLDER's inequality implies that every square integrable function $v : J \longrightarrow \overline{\mathbb{R}}$ is LEBESGUE integrable with

$$0 \;\leq\; \int_J |v(s)| \; d\mathcal{L}^1 s \;\leq\; \sqrt{\int_J 1 \; d\mathcal{L}^1 s} \cdot \sqrt{\int_J |v(s)|^2 \; d\mathcal{L}^1 s} \;<\; \infty .$$

In short, $L^2(J) \subset L^1(J)$ holds for any bounded interval $J \subset \mathbb{R}$.

Now we combine the preceding choice of absolutely continuous functions with the stronger condition of square integrable weak derivatives:

Definition 2.9 *For any nonempty compact interval $J \subset \mathbb{R}$, we define*

$$W^{1,2}(J) \;:=\; \Big\{ v \in AC(J, \mathbb{R}) \;\Big|\; v \text{ and its weak derivative } v' \text{ are in } L^2(J) \Big\},$$
$$W_0^{1,2}(J) \;:=\; \Big\{ v \in W^{1,2}(J) \;\Big|\; \forall\, s \in \partial J : \;\; v(s) = 0 \Big\}.$$

$W^{1,2}(J)$ *is called a* <u>SOBOLEV space (of order 1 and exponent 2)</u>.

Lemma 2.10 *Let $J \subset \mathbb{R}$ be a nonempty compact interval.*
Then $W^{1,2}(J)$ and $W_0^{1,2}(J)$ supplied with pointwise addition and scalar multiplication are real vector spaces.

Proof: Choose $\lambda \in \mathbb{R}$ and any $u, v \in W^{1,2}(J) \subset AC(J)$. Let $u', v' : J \longrightarrow \mathbb{R}$ respectively denote their weak derivatives. We obtain for any $t_0, t_1 \in J$ $(t_0 < t_1)$

$$\begin{aligned} \lambda\, u(t_1) + v(t_1) \;-\; \big(\lambda\, u(t_0) + v(t_0)\big) \;&=\; \lambda \int_{[t_0,\, t_1]} u' \; d\mathcal{L}^1 s \;+\; \int_{[t_0,\, t_1]} v' \; d\mathcal{L}^1 s \\ &=\; \int_{[t_0,\, t_1]} \big(\lambda \;\, u'(s) \;+\; v'(s)\big) \;\, d\mathcal{L}^1 s, \end{aligned}$$

i.e., the LEBESGUE integrable function $\lambda\, u' + v' : J \longrightarrow \mathbb{R}$ is the weak derivative of the LEBESGUE measurable function $\lambda \;\, u(\cdot) + v(\cdot)$. Hence, $\lambda \;\, u(\cdot) + v(\cdot)$ is absolutely continuous. Furthermore, $L^2(J)$ is known to be a vector space and so, $u, u', v, v' \in L^2(J)$ implies that both $\lambda\; u(\cdot) + v(\cdot)$ and $\lambda\; u'(\cdot) + v'(\cdot)$ belong to $L^2(J)$, i.e., $\lambda\, u + v \in W^{1,2}(J)$. If $u(\cdot)$ and $v(\cdot)$ have all boundary values $= 0$ so has $\lambda\; u(\cdot) + v(\cdot)$. □

Now we have all the tools needed for formulating a generalised problem: In the proof of DIRICHLET's principle (on page 5 ff.), we characterized the minimizer $u \in C_0^0([0,1]) \cap C^1(]0,1[)$ of $\Psi(\cdot)$ as a solution of the variational equation

$$\int_0^1 \Big(u'(t) \cdot \varphi'(t) \;+\; f(t) \cdot \varphi(t)\Big) \; dt \;=\; 0$$

for every test function $\varphi \in C_0^0([0,1]) \cap C^1(]0,1[)$. In the generalised setting, $W_0^{1,2}([0,1])$ is to play the role of $C_0^0([0,1]) \cap C^1(]0,1[)$ and, all derivatives are interpreted in the weak sense of Definition 2.3 (on page 11). This notion leads to a new concept of solution:

Definition 2.11 *Consider the boundary value problem*

$$\wedge \begin{cases} u''(t) &= f(t) \quad \textit{for every } t \in]0,1[\\ u(0) &= 0 \\ u(1) &= 0. \end{cases}$$

For any mapping $f \in L^2([0,1])$ given, a function $u \in W_0^{1,2}([0,1])$ is called its <u>weak solution</u> *if*

$$\int_{[0,1]} u'(t) \cdot \varphi'(t) \;\, d\mathcal{L}^1 t \;+\; \int_{[0,1]} f(t) \cdot \varphi(t) \;\, d\mathcal{L}^1 t \;=\; 0 \qquad (**)$$

holds for every test function $\varphi \in W_0^{1,2}([0,1])$.

The search for a weak solution starts with the (quite simple, but important) observation that in criterion $(**)$, only one of the two LEBESGUE integrals takes the wanted function $u(\cdot)$ into consideration.
Furthermore, criterion $(**)$ can be regarded as a comparison of two functions, namely

$$W_0^{1,2}([0,1]) \longrightarrow \mathbb{R}, \qquad \varphi \longmapsto \int_{[0,1]} u'(t) \cdot \varphi'(t) \;\, d\mathcal{L}^1 t,$$

$$W_0^{1,2}([0,1]) \longrightarrow \mathbb{R}, \qquad \varphi \longmapsto -\int_{[0,1]} f(t) \cdot \varphi(t) \;\, d\mathcal{L}^1 t.$$

Both of them are linear and, the second one is continuous with respect to the $L^2(\mathbb{R})$ norm because HÖLDER's inequality (Lemma 2.6 on page 16) implies

$$\left| - \int_{[0,1]} f(t) \cdot \varphi(t) \;\; d\mathcal{L}^1 t \right| \;\; \leq \;\; \|f\|_{L^2([0,1])} \;\; \|\varphi\|_{L^2([0,1])} \, .$$

In the next step, we focus on the LEBESGUE integral involving the wanted function $u \in W_0^{1,2}([0,1])$. It induces a mapping of *two* SOBOLEV functions:

$$\begin{aligned} \langle \cdot, \, \cdot \rangle_{W_0^{1,2}([0,1])} \;:\;\; W_0^{1,2}([0,1]) \times W_0^{1,2}([0,1]) \;\; &\longrightarrow \;\; \mathbb{R}, \\ (u, \; v) \;\; &\longmapsto \;\; \int_{[0,1]} u'(t) \cdot v'(t) \;\; d\mathcal{L}^1 t. \end{aligned}$$

2.v The Hilbert space $W_0^{1,2}([0,1])$

The Lebesgue integral is known to be linear and so, we obtain:

Lemma 2.12

$$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle_{W_0^{1,2}([0,1])} \;:\; W_0^{1,2}([0,1]) \times W_0^{1,2}([0,1]) &\longrightarrow \mathbb{R},\\ (u,\ v) &\longmapsto \int_{[0,1]} u'(t)\cdot v'(t)\ d\mathcal{L}^1 t\end{aligned}$$

is a symmetric bilinear form, i.e., for all $u, v, w \in W_0^{1,2}([0,1])$ *and* $\lambda \in \mathbb{R}$,

$$\wedge \begin{cases} \langle u,\ v\rangle_{W_0^{1,2}([0,1])} &= \langle v,\ u\rangle_{W_0^{1,2}([0,1])}\\ \langle \lambda\, u + v,\ w\rangle_{W_0^{1,2}([0,1])} &= \lambda\,\langle u,\ w\rangle_{W_0^{1,2}([0,1])} + \langle v,\ w\rangle_{W_0^{1,2}([0,1])}\,. \end{cases}$$

If $\langle\,\cdot\,,\,\cdot\,\rangle_{W_0^{1,2}([0,1])}$ proves to be positive definite in addition, then it is even an inner product on $W_0^{1,2}([0,1])$. The so-called Poincaré's inequality is the key tool:

Lemma 2.13 (Poincaré's inequality in one variable)
Consider $J = [a,b] \subset \mathbb{R}$ *with* $a < b$. *Then every* $u \in W_0^{1,2}(J)$ *satisfies*

$$\int_J |u|^2\ d\mathcal{L}^1 s \;\le\; \tfrac{(b-a)^2}{2}\cdot\int_J |u'|^2\ d\mathcal{L}^1 s\,.$$

Proof: Every $u \in W_0^{1,2}(J)$ is absolutely continuous by definition and so, we obtain for every $t \in J$

$$u(t) \;=\; u(a) \;+\; \int_{[0,t]} u'(s)\ d\mathcal{L}^1 s \;=\; 0 \;+\; \int_{[0,t]} u'(s)\ d\mathcal{L}^1 s.$$

Hölder's inequality (Lemma 2.6 on page 16) implies for every $t \in J$

$$\begin{aligned} |u(t)| \;=\; \left|\int_{[a,t]} u'(s)\ d\mathcal{L}^1 s\right| &\le \sqrt{\int_{[a,t]} 1\ d\mathcal{L}^1 s}\cdot\sqrt{\int_{[a,t]} |u'(s)|^2\ d\mathcal{L}^1 s}\\ &\le \sqrt{t-a}\cdot\sqrt{\int_J |u'(s)|^2\ d\mathcal{L}^1 s}\\ \Longrightarrow\qquad |u(t)|^2 &\le (t-a)\cdot\|u'\|^2_{L^2(J)}\,. \end{aligned}$$

Integrating with respect to t leads to the claimed estimate. □

Corollary 2.14 *$\langle \cdot, \cdot \rangle_{W_0^{1,2}([0,1])}$ is an inner product on $W_0^{1,2}([0,1])$ and thus,*

$$\| \cdot \|_{W_0^{1,2}([0,1])} : \quad W_0^{1,2}([0,1]) \longrightarrow \mathbb{R}, \qquad u \longmapsto \|u\|_{W_0^{1,2}([0,1])} := \sqrt{\langle u, u \rangle_{W_0^{1,2}([0,1])}}$$

is a norm. □

The final step which we need for applying the tools of functional analysis later on concerns the completeness.

Proposition 2.15 *$\left(W_0^{1,2}([0,1]), \ \langle \cdot, \cdot \rangle_{W_0^{1,2}([0,1])}\right)$ is a* HILBERT *space, i.e., a complete normed vector space whose norm is induced by an inner product.*

Proof: It remains to prove that every CAUCHY sequence $(u_k)_{k\in\mathbb{N}}$ in the normed vector space $\left(W_0^{1,2}([0,1]), \ \| \cdot \|_{W_0^{1,2}([0,1])}\right)$ has a limit in $W_0^{1,2}([0,1])$. We apply the FISCHER–RIESZ theorem stating that $\left(L^2([0,1]), \ \| \cdot \|_{L^2([0,1])}\right)$ is complete.

Whenever $(u_k)_{k\in\mathbb{N}}$ is a CAUCHY sequence with respect to $\| \cdot \|_{W_0^{1,2}([0,1])}$ then the sequence $(u_k')_{k\in\mathbb{N}}$ of weak derivatives is a CAUCHY sequence in $L^2([0,1])$ and so, there exists $w \in L^2([0,1])$ with

$$\| u_k' - w \|_{L^2([0,1])} \longrightarrow 0 \qquad \text{for } k \longrightarrow \infty.$$

Furthermore, POINCARÉ's inequality (Lemma 2.13) states

$$\| u_j - u_k \|_{L^2([0,1])} \le \sqrt{2} \cdot \| u_j' - u_k' \|_{L^2([0,1])} \overset{\text{Def.}}{=} \sqrt{2} \cdot \| u_j - u_k \|_{W_0^{1,2}([0,1])}$$

for any indices j, k and thus, $(u_k)_{k\in\mathbb{N}}$ is also a CAUCHY sequence in $L^2([0,1])$. The FISCHER–RIESZ theorem ensures some $v \in L^2([0,1])$ with

$$\| u_k - v \|_{L^2([0,1])} \longrightarrow 0 \qquad \text{for } k \longrightarrow \infty.$$

Now we verify that $w(\cdot)$ is the weak derivative of $v(\cdot)$. Due to Proposition 2.5 (on page 11), every test function $\psi \in C^1([0,1]) \cap C_0^0([0,1])$ and each index $k \in \mathbb{N}$ fulfil

$$\int_{[0,1]} u_k(s) \cdot \psi'(s) \; d\mathcal{L}^1 s \;\; = \;\; - \int_{[0,1]} u_k'(s) \cdot \psi(s) \; d\mathcal{L}^1 s.$$

The limit for $k \longrightarrow \infty$ reveals by means of HÖLDER's inequality

$$\int_{[0,1]} v(s) \cdot \psi'(s) \; d\mathcal{L}^1 s \;\; = \;\; - \int_{[0,1]} w(s) \cdot \psi(s) \; d\mathcal{L}^1 s.$$

This feature for every $\psi \in C^1([0,1]) \cap C_0^0([0,1])$ characterizes $w \in L^2([0,1]) \subset L^1([0,1])$ as the weak derivative of $v(\cdot)$.

Hence, $v(\cdot)$ belongs to the SOBOLEV space $W^{1,2}([0,1])$ with $v' = w$ and

$$\|u_k - v\|_{L^2([0,1])} + \|u_k' - v'\|_{L^2([0,1])} \longrightarrow 0 \qquad \text{for } k \longrightarrow \infty.$$

Finally, it remains to prove $v \in W_0^{1,2}([0,1])$, i.e., the limit function $v(\cdot)$ satisfies $v(0) = 0 = v(1)$ in addition. Due to $W^{1,2}([0,1]) \subset AC([0,1]) \subset C^0([0,1])$, $v(\cdot)$ is continuous in $[0,1]$. Furthermore, $(u_k)_{k\in\mathbb{N}}$ converges to $v(\cdot)$ with respect to $\|\cdot\|_{L^2([0,1])}$ and so, there exist a subsequence $(u_{k_m})_{m\in\mathbb{N}}$ and a LEBESGUE measurable subset $\widetilde{J} \subset [0,1]$ with

$$\wedge \begin{cases} \lim\limits_{m\to\infty} u_{k_m}(t) \; = \; v(t) & \text{for every } t \in \widetilde{J}, \\ \mathcal{L}^1([0,1] \setminus \widetilde{J}) \; = \; 0. \end{cases}$$

Fixing any $t_0 \in \widetilde{J}$, we obtain for every index $m \in \mathbb{N}$

$$\begin{aligned} u_{k_m}(t_0) \;&=\; u_{k_m}(0) + \int_{[0,\,t_0]} u_{k_m}'(s) \; d\mathcal{L}^1 s \\ &=\; \quad 0 \quad + \int_{[0,\,t_0]} u_{k_m}'(s) \; d\mathcal{L}^1 s \end{aligned}$$

due to $u_{k_m} \in W_0^{1,2}([0,1])$. For $m \longrightarrow \infty$, both sides of the equation converge

$$v(t_0) \;\; = \;\; \int_{[0,\,t_0]} v'(s) \; d\mathcal{L}^1 s.$$

The absolute continuity of $v(\cdot)$ implies $v(0) \; = \; v(t_0) - \int_{[0,\,t_0]} v'(s) \; d\mathcal{L}^1 s \; = \; 0.$ The remaining claim $v(1) = 0$ can be verified in an analogous way. □

3 Some Boundary Value Problems for Partial Diff. Equations

It has already been admitted frankly in § 2.I that the boundary value problem

$$\wedge \begin{cases} u''(t) &= f(t) \quad \text{for every } t \in \,]0,1[\\ u(0) &= 0 \\ u(1) &= 0 \end{cases}$$

is rather a toy example since integration leads to its explicit solution. We more or less regard it as a starting point for a new solution concept in the field of differential equations, i.e., *weak* solutions in a SOBOLEV space.
This approach can be extended in various directions, of course. A possibility is to investigate more complicated (but still) ordinary differential equations of second order with DIRICHLET boundary conditions. In many cases, we are then free to interpret it as a fixed point problem ...

In this section, however, we continue in another direction: We focus on *partial* differential equations of second order (instead of ordinary ones) ... and their DIRICHLET boundary value problems. POISSON*'s equation* also called *inhomogeneous* LAPLACE*'s equation* is the well-established counterpart of our ODE example, which is used frequently in physics and mechanics:

$$\wedge \begin{cases} \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

with a nonempty open set $\Omega \subset \mathbb{R}^n$ and a function $f : \Omega \longrightarrow \mathbb{R}$ given. In particular, the domain $\Omega \subset \mathbb{R}^n$ is not restricted to examples of simple geometry like squares or tubes and so, many standard techniques for explicit solutions (such as the separation of variables or explicit GREEN's functions) fail now.

3.I DIRICHLET's Principle for POISSON's Equation

The good news is that DIRICHLET's principle also holds for more than just one independent variable, i.e., Proposition 2.2 (on page 5) can be generalised to POISSON's equation with zero DIRICHLET boundary conditions.

We just have to adapt our analytical tools to functions of several real variables. In the proof of Proposition 2.2 and Definition 2.11 of a weak solution, integration by parts plays a key role. Its counterpart in multivariable calculus consists in GAUSS' theorem and its consequences, i.e., GREEN's formulas.
For the sake of the simplicity, we restrict our considerations to *zero* DIRICHLET boundary conditions. As first advantage, we then need only a very simple form of GAUSS' theorem:

Definition 3.1 *Let $\Omega \subset \mathbb{R}^n$ be an nonempty open set.*
The support *of a function $g : \Omega \longrightarrow \mathbb{R}$ is defined as*

$$\text{supp } g \ := \ \overline{\left\{ x \in \Omega \;\middle|\; g(x) \neq 0 \right\}} \ \subset \ \mathbb{R}^n .$$

A function $g : \Omega \longrightarrow \mathbb{R}$ is said to have compact support (in Ω) *if the subset* supp $g \subset \mathbb{R}^n$ *is compact and contained in Ω.*
$C_c^0(\Omega)$ and $C_c^k(\Omega)$ $(k \in \mathbb{N})$ denote the sets of functions $\Omega \longrightarrow \mathbb{R}$ with compact support which are continuous and k times continuously differentiable respectively.

The fundamental theorem of calculus and successive integration with respect to the real components imply the following special case of GAUSS' theorem:

Proposition 3.2 (Special case of GAUSSS' theorem for compact support)
Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set and $j \in \{1, \ldots, n\}$. Every continuously differentiable function $g : \Omega \longrightarrow \mathbb{R}$ with compact support satisfies

$$\int_\Omega \partial_{x_j} g(x) \ d\mathcal{L}^n x \ = \ 0. \qquad \square$$

Corollary 3.3 (GREEN's formulas for compact support)
Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set and $u \in C^1(\Omega)$, $v \in C_c^1(\Omega)$. Then

$$(1.) \quad \int_\Omega \partial_{x_j} u \cdot v \ \ d\mathcal{L}^n x \ = \ - \int_\Omega u \cdot \partial_{x_j} v \ \ d\mathcal{L}^n x \quad \text{for each } j \in \{1, \ldots, n\},$$

$$(2.) \quad \int_\Omega \Delta u \cdot v \ \ d\mathcal{L}^n x \ = \ \int_\Omega u \cdot \Delta v \ \ d\mathcal{L}^n x \quad \text{whenever } u \in C^2(\Omega), v \in C_c^2(\Omega).$$

Formula (1.) is the obvious extension of integration by parts since all boundary values are zero. That is essentially all we need for adapting DIRICHLET's principle:

Proposition 3.4 (DIRICHLET's principle for POISSON's equation)
Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set and $f \in C^0(\Omega)$. Suppose that

$$\begin{aligned} \Psi : \ C_c^1(\Omega) \ &\longrightarrow \ \mathbb{R}, \\ v(\cdot) \ &\longmapsto \ \tfrac{1}{2} \cdot \int_\Omega \|\nabla v(x)\|^2 \ d\mathcal{L}^n x \ + \ \int_\Omega f(x) \cdot v(x) \ d\mathcal{L}^n x \end{aligned}$$

has its global minimum at $u(\cdot)$. Furthermore assume $u \in C^2(\Omega)$.

Then the trivial extension of this minimizer $u : \Omega \longrightarrow \mathbb{R}$ to the closure $\overline{\Omega}$ is a solution to the boundary value problem

$$\wedge \begin{cases} \Delta u \ = \ f & \text{in } \Omega \\ u \ = \ 0 & \text{on } \partial\Omega. \end{cases}$$

The proof is based on exactly the same arguments as for Proposition 2.2 – just using now the extensions to functions of several variables. For the sake of transparency, we present it in detail for underlining the analogy due to LEBESGUE's theory of integration.

Proof: Fix an auxiliary function $\varphi \in C_c^1(\Omega)$ arbitrarily. For each $h \in \mathbb{R}$,

$$v_h \ := \ u(\cdot) + h \cdot \varphi(\cdot) : \ \Omega \ \longrightarrow \ \mathbb{R}, \qquad x \ \longmapsto \ u(x) + h \cdot \varphi(x)$$

is continuously differentiable, has compact support in Ω and so, it belongs to the domain of Ψ. Since Ψ has its global minimum at $u(\cdot)$, the composition

$$\mathbb{R} \longrightarrow \mathbb{R}, \qquad h \longmapsto \Psi(v_h) \overset{\text{Def.}}{=} \Psi(u + h \cdot \varphi)$$

attains its global minimum at $h = 0$:

$$\frac{1}{2}\int_\Omega \|\nabla v_h\|^2 \, d\mathcal{L}^n x + \int_\Omega f \cdot v_h \, d\mathcal{L}^n x \;\geq\; \frac{1}{2}\int_\Omega \|\nabla u\|^2 \, d\mathcal{L}^n x + \int_\Omega f \cdot u \, d\mathcal{L}^n x$$

for all $h \in \mathbb{R}$. We conclude for every $h \in \mathbb{R}$

$$\begin{aligned}
& \int_\Omega \Big(\tfrac{1}{2} \|\nabla v_h\|^2 - \tfrac{1}{2} \|\nabla u\|^2 \;+\; f \cdot (v_h - u) \Big) \; d\mathcal{L}^n x \;\geq\; 0 \\
\Longleftrightarrow \quad & \int_\Omega \Big(\tfrac{1}{2} \|\nabla u + h\, \nabla \varphi\|^2 - \tfrac{1}{2} \|\nabla u\|^2 \;+\; f \cdot \, h \cdot \varphi \Big) \; d\mathcal{L}^n x \;\geq\; 0 \\
\Longleftrightarrow \quad & \int_\Omega \Big(h \, \langle \nabla u, \, \nabla \varphi \rangle_{\mathbb{R}^n} \;+\; \tfrac{1}{2}\, h^2 \cdot \|\nabla \varphi\|^2 \;+\; f \cdot \, h \cdot \varphi \Big) \; d\mathcal{L}^n x \;\geq\; 0 \\
\Longleftrightarrow \quad & \tfrac{1}{2}\, h^2 \cdot \int_\Omega \|\nabla \varphi\|^2 \; d\mathcal{L}^n x \;+\; h \cdot \int_\Omega \Big(\langle \nabla u, \, \nabla \varphi \rangle_{\mathbb{R}^n} \;+\; f \cdot \varphi \Big) \; d\mathcal{L}^n x \;\geq\; 0.
\end{aligned}$$

Restricting our considerations to any $h > 0$, in particular, we obtain

$$\tfrac{1}{2}\, h \cdot \int_\Omega \|\nabla \varphi\|^2 \; d\mathcal{L}^n x \;+\; \int_\Omega \Big(\langle \nabla u, \, \nabla \varphi \rangle_{\mathbb{R}^n} \;+\; f \cdot \varphi \Big) \; d\mathcal{L}^n x \;\;\geq\;\; 0.$$

The limit for $h \longrightarrow 0^+$ leads to the necessary condition

$$\int_\Omega \Big(\langle \nabla u, \, \nabla \varphi \rangle_{\mathbb{R}^n} \;+\; f \cdot \varphi \Big) \; d\mathcal{L}^n x \;\;\geq\;\; 0$$

for every $\varphi \in C^1_c(\Omega)$. Applying this inequality to both φ and $-\varphi$, we even conclude the equality for every $\varphi \in C^1_c(\Omega)$

$$\int_\Omega \Big(\langle \nabla u, \, \nabla \varphi \rangle_{\mathbb{R}^n} \;+\; f \cdot \varphi \Big) \; d\mathcal{L}^n x \;\;=\;\; 0. \qquad (*)$$

Due to $u \in C^2(\Omega)$, GREEN's formula (Proposition 3.3 (1.)) leads to

$$\begin{aligned}
& \int_\Omega \Big(- \, \Delta u \cdot \varphi \;+\; f \cdot \varphi \Big) \; d\mathcal{L}^n x \;\;=\;\; 0 \\
\Longleftrightarrow \quad & \int_\Omega \Big(- \, \Delta u \;+\; f \Big) \cdot \varphi \; d\mathcal{L}^n x \;\;=\;\; 0
\end{aligned}$$

for every $\varphi \in C^1_c(\Omega)$. As an indirect consequence of continuity, this variational equation can hold only if $-\Delta u + f \;=\; 0$ in Ω. □

3.II The HILBERT space $W_0^{1,2}(\Omega)$

The next goal is to adapt the concept of weak solutions to POISSON's equations. In particular, we need weak derivatives for functions of several variables. It appears inevitable that the analytical details become more technical ...

Proposition 2.5 (3.) characterizing absolutely continuous functions (on page 11) differs from GREEN's formula in Proposition 3.3 (1.) just in regard to number of real variables. The different dimensions of the LEBESGUE integrals are immediate consequences. Hence we use this similarity as starting point for a definition:

Definition 3.5 *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set, $j \in \{1, \ldots, n\}$ and $f, g : \Omega \longrightarrow \mathbb{R}$ be locally* LEBESGUE *integrable.*

$g(\cdot)$ *is called the* weak partial derivative *of $f(\cdot)$* with respect to the variable x_j *if the following condition is satisfied for every $\psi \in C_c^1(\Omega)$*

$$\int_\Omega f(x) \cdot \partial_{x_j}\psi(x) \;\, d\mathcal{L}^n x \;\; = \;\; - \int_\Omega g(x) \cdot \psi(x) \;\, d\mathcal{L}^n x\,.$$

It is usually denoted as $\partial_j f := \partial_{x_j} f := g$.

The gist of this definition is often summarized in the short statement: The rule of integration by parts still holds – formally, at least. Strictly speaking, it is one of the GREEN's formulas. "Formally" here refers to the two details that we consider weak derivatives (instead of the classical ones) and minimal regularity assumptions have substituted the condition of continuity which comes from the fundamental theorem of calculus.

"Local" integrability in the sense of LEBESGUE means that the considered function is LEBESGUE integrable over any compact subset of $\Omega \subset \mathbb{R}^n$ (but not necessarily over the whole set Ω, which might not be bounded). This weaker feature is completely sufficient here because $\psi \in C_c^1(\Omega)$ implies the compact support of both products $f \cdot \partial_{x_j}\psi$ and $g \cdot \psi$ and so their LEBESGUE integrability.

The step to so-called SOBOLEV spaces is based on the idea that both the function $f : \Omega \longrightarrow \mathbb{R}$ and its weak derivatives $\partial_{x_1} f, \ \ldots, \ \partial_{x_n} f$ belong to one and the same LEBESGUE space, namely $L^2(\Omega)$ here.
The well-known inner product of $L^2(\Omega)$

$$L^2(\Omega) \times L^2(\Omega) \ \longrightarrow \ \mathbb{R}, \quad (f, g) \ \longmapsto \ \langle f, \, g \rangle_{L^2(\Omega)} := \int_\Omega f(x) \cdot g(x) \ d\mathcal{L}^n x$$

induces a bilinear form on the SOBOLEV space by taking both the functions and all their weak derivatives into consideration:

Definition 3.6 *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set.*

The SOBOLEV *space $W^{1,2}(\Omega)$ consists of all functions $f : \Omega \longrightarrow \mathbb{R}^n$ satisfying*

- *$f(\cdot)$ is square integrable $\big(f \in L^2(\Omega)\big)$, i.e., $|f|^2$ is* LEBESGUE *integrable,*
- *$f(\cdot)$ has a weak derivative $\partial_j f$ w.r.t. each variable x_j, $j \in \{1, \ldots n\}$,*
- *$\partial_j f$ is square integrable, i.e., $\partial_j f \in L^2(\Omega)$ for every $j \in \{1, \ldots n\}$.*

It is supplied with the inner product

$$\langle f, g \rangle_{W^{1,2}(\Omega)} \ := \ \int_\Omega f \cdot g \ \mathcal{L}^n x \ + \ \sum_{j=1}^{n} \int_\Omega \partial_{x_j} f \cdot \partial_{x_j} g \ \mathcal{L}^n x$$

and thus with the induced SOBOLEV *norm $\| \cdot \|_{W^{1,2}(\Omega)} \ := \ \sqrt{\langle \cdot , \cdot \rangle_{W^{1,2}(\Omega)}}$.*

According to the well-known theorem of FISCHER–RIESZ, $(L^2(\Omega), \ \| \cdot \|_{L^2(\Omega)})$ is complete. As a consequence, $(W^{1,2}(\Omega), \ \| \cdot \|_{W^{1,2}(\Omega)})$ is also complete since the key arguments in proof of Proposition 2.15 (on page 21) are easy to extend to functions of several variables.

Corollary 3.7 *For any nonempty open set $\Omega \subset \mathbb{R}^n$, $(W^{1,2}(\Omega), \ \langle \cdot, \cdot \rangle_{W^{1,2}(\Omega)})$ is a real* HILBERT *space.*

Finally we want to generalise zero DIRICHLET boundary conditions.
In this step, we have to deal with the following two challenges in particular:

- By definition, functions in $W^{1,2}(\Omega)$ are characterized just in terms of square LEBESGUE integrable functions on Ω and, we are not aware of any relations to continuous functions on Ω or even $\overline{\Omega}$ so far. Hence it is not clear how to specify boundary values of $f \in W^{1,2}(\Omega) \subset L^2(\Omega)$.

- We would like to give a criterion for the zero DIRICHLET boundary condition *without* additional restrictions on the open set $\Omega \subset \mathbb{R}^n$.

In connections with smooth functions, the compact support has already proved to be a very useful assumption. It ensures particularly that not just the function under consideration is identical to 0 close to the boundary of the domain, but so are all its derivatives. This is completely independent of the regularity of $\partial\Omega$ and motivated us to introduce $C_c^k(\Omega)$ for every order $k \in \mathbb{N}$ in Definition 3.1 (on page 24).
Interpreting now $C_c^1(\Omega)$ as the "classical" class of differentiable functions fulfilling zero DIRICHLET boundary conditions, those SOBOLEV functions in $W^{1,2}(\Omega)$ which can be approximated by a sequence in $C_c^1(\Omega)$ are nominated as generalisations. This is not the only choice possible in functional analysis, of course, but it has proved to be a quite convenient one and thus, it has become well established in the literature.

Definition 3.8 *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set.*
$W_0^{1,2}(\Omega)$ abbreviates the closure of $C_c^1(\Omega)$ w.r.t. the SOBOLEV *norm* $\|\cdot\|_{W^{1,2}(\Omega)}$:
$$W_0^{1,2}(\Omega) := \left\{ f \in W^{1,2}(\Omega) \,\middle|\, \exists\, (\varphi_k)_{k\in\mathbb{N}} \text{ in } C_c^1(\Omega) : \lim_{k\to\infty} \|f-\varphi_k\|_{W^{1,2}(\Omega)} = 0 \right\}.$$

This opens the door to introducing weak solutions to POISSON's equation with zero DIRICHLET boundary conditions – similarly to Definition 2.11 for functions of one variable (on page 18) and the variational equation $(*)$ in the proof of DIRICHLET's principle 3.4 (on page 26) for functions of several variables:

Definition 3.9 *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set and consider the boundary value problem*

$$\wedge \begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For any mapping $f \in L^2(\Omega)$ given, a function $u \in W_0^{1,2}(\Omega)$ is called its <u>weak solution</u> *if*

$$\int_\Omega \langle \nabla u(x), \nabla \varphi(x) \rangle_{\mathbb{R}^n} \; d\mathcal{L}^n x \; + \; \int_\Omega f(x) \cdot \varphi(x) \; d\mathcal{L}^n x \; = \; 0$$

holds for every test function $\varphi \in W_0^{1,2}(\Omega)$.

The wanted SOBOLEV function $u \in W_0^{1,2}(\Omega)$ occurs just in the first LEBESGUE integral inducing the bilinear form

$$W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \longrightarrow \mathbb{R}, \quad (u, v) \longmapsto \int_\Omega \langle \nabla u(x), \nabla v(x) \rangle_{\mathbb{R}^n} \; d\mathcal{L}^n x.$$

In the special case of one variable (in § 2), we have already observed that this bilinear form is positive definite and thus an inner product – due to POINCARÉ's inequality in Lemma 2.13 (on page 20). This phenomenon also occurs for functions of more than one variable:

Proposition 3.10 (POINCARÉ's inequality for $W_0^{1,2}(\Omega)$)
Suppose the nonempty open subset $\Omega \subset \mathbb{R}^n$ to be bounded.
Then there exists a constant $C = C(\Omega) > 0$ satisfying for every $u \in W_0^{1,2}(\Omega)$

$$\int_\Omega |u(x)|^2 \; \mathcal{L}^n x \; \leq \; C \cdot \int_\Omega \|\nabla u(x)\|^2 \; \mathcal{L}^n x.$$

Proof: $\Omega \subset \mathbb{R}^n$ is assumed to be bounded and so, there exists a radius $R > 0$ with $\Omega \subset \;]-R, R[^n \subset \mathbb{R}^n$. Set $Q := \;]-R, R[^n \subset \mathbb{R}^n$. We are going to verify the constant $C = 4\,R^2 > 0$ in two steps:

Step 1: The claimed inequality for "smooth" $u(\cdot)$ with compact support.

Choose any $u \in C_c^1(\Omega)$. Then the trivial extension of $u(\cdot)$ to $[-R, R]^n$

$$\widetilde{u}: \quad [-R, R]^n \longrightarrow \mathbb{R}, \qquad x \longmapsto \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in [-R, R]^n \setminus \Omega \end{cases}$$

is continuously differentiable and, all its boundary values are 0.

The fundamental theorem of calculus and HÖLDER's inequality (Lemma 2.6 on page 16) ensure for every $x \in [-R, R]^n \subset \mathbb{R}^n$

$$\begin{aligned}
& \widetilde{u}(x) && = \int_{-R}^{x_j} \partial_j \widetilde{u}\Big(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots x_n\Big) \; d\xi \\
\Longrightarrow \quad & |\widetilde{u}(x)| && \leq \int_{-R}^{x_j} \Big|\partial_j \widetilde{u}\Big(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots x_n\Big)\Big| \; d\xi \\
\Longrightarrow \quad & |\widetilde{u}(x)| && \leq \sqrt{\int_{-R}^{x_j} 1 \; d\xi} \; \sqrt{\int_{-R}^{x_j} \Big|\partial_j \widetilde{u}|_{(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots x_n)}\Big|^2 \; d\xi} \\
\Longrightarrow \quad & |\widetilde{u}(x)|^2 && \leq (x_j + R) \int_{-R}^{x_j} \Big|\partial_j \widetilde{u}|_{(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots x_n)}\Big|^2 \; d\xi \\
\Longrightarrow \quad & |\widetilde{u}(x)|^2 && \leq 2R \int_{-R}^{R} \Big|\partial_j \widetilde{u}|_{(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots x_n)}\Big|^2 \; d\xi \, .
\end{aligned}$$

Next we integrate this inequality with respect to $x_j \in [-R, R]$

$$\begin{aligned}
\int_{-R}^{R} |\widetilde{u}|^2 \; dx_j & \leq \int_{-R}^{R} \Big(2R \cdot \int_{-R}^{R} \Big|\partial_j \widetilde{u}|_{(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots x_n)}\Big|^2 \; d\xi\Big) \; dx_j \\
& = (2R)^2 \int_{-R}^{R} \Big|\partial_j \widetilde{u}|_{(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots x_n)}\Big|^2 \; d\xi \, .
\end{aligned}$$

Finally the successive integration with respect to $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots x_n$ leads to

$$\begin{aligned}
& \int_{[-R, R]^n} |\widetilde{u}(x)|^2 \; dx && \leq (2R)^2 \int_{[-R, R]^n} |\partial_j \widetilde{u}(x)|^2 \; dx \\
\Longrightarrow \quad & \int_{\Omega} |u(x)|^2 \; d\mathcal{L}^n x && \leq (2R)^2 \int_{\Omega} |\partial_j u(x)|^2 \; d\mathcal{L}^n x \\
& && \leq (2R)^2 \int_{\Omega} \|\nabla u(x)\|^2 \; d\mathcal{L}^n x \, .
\end{aligned}$$

Step 2: Extending the inequality to any $u \in W_0^{1,2}(\Omega)$.

For each $u \in W_0^{1,2}(\Omega)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $C_c^1(\Omega)$ satisfying

$$\|u - \varphi_k\|^2_{W^{1,2}(\Omega)} \overset{\text{Def.}}{=} \|u - \varphi_k\|^2_{L^2(\Omega)} + \sum_{j=1}^{n} \|\partial_j u - \partial_j \varphi_k\|^2_{L^2(\Omega)} \overset{k\to\infty}{\longrightarrow} 0$$

due to Definition 3.8. The first step implies for each $k \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$

$$\begin{aligned}
\|\varphi_k\|^2_{L^2(\Omega)} \overset{\text{Def.}}{=} \int_\Omega |\varphi_k(x)|^2 \; d\mathcal{L}^n x &\le (2\,R)^2 \cdot \int_\Omega |\partial_j\,\varphi_k(x)|^2 \; d\mathcal{L}^n x \\
&\overset{\text{Def.}}{=} (2\,R)^2 \cdot \|\partial_j\,\varphi_k\|^2_{L^2(\Omega)} \\
\Longrightarrow \qquad \|\varphi_k\|_{L^2(\Omega)} &\le 2\,R \cdot \|\partial_j\,\varphi_k\|_{L^2(\Omega)}\,.
\end{aligned}$$

The limit for $k \longrightarrow \infty$ reveals

$$\begin{aligned}
\|u\|_{L^2(\Omega)} = \lim_{k\to\infty} \|\varphi_k\|_{L^2(\Omega)} &\le 2\,R \cdot \lim_{k\to\infty} \|\partial_j\,\varphi_k\|_{L^2(\Omega)} \\
&= 2\,R \cdot \|\partial_j\,u\|_{L^2(\Omega)}\,.
\end{aligned}$$

□

In our context, the main benefit of POINCARÉ's inequality is the following result:

Corollary 3.11 *Let the nonempty set $\Omega \subset \mathbb{R}^n$ be open and bounded. Then the bilinear form*

$$\begin{aligned}
\langle\cdot,\cdot\rangle_{W_0^{1,2}(\Omega)} : \quad W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) &\longrightarrow \mathbb{R}, \\
(u, v) &\longmapsto \int_\Omega \langle \nabla u,\ \nabla v\rangle_{\mathbb{R}^n} \; d\mathcal{L}^n x
\end{aligned}$$

is positive definite and so an inner product on $W_0^{1,2}(\Omega)$. □

Finally, we conclude essentially from the completeness of $\big(W^{1,2}(\Omega), \|\cdot\|_{W^{1,2}(\Omega)}\big)$ stated in Corollary 3.7 (on page 28):

Corollary 3.12 *Let the nonempty set $\Omega \subset \mathbb{R}^n$ be bounded and open. Then, $\Big(W_0^{1,2}(\Omega),\ \langle\cdot,\cdot\rangle_{W_0^{1,2}(\Omega)}\Big)$ is a real* HILBERT *space.*

4 The General Linear Problem in a Real HILBERT Space

4.I RIESZ' Representation Theorem

Recently Definition 2.11 of a weak solution to a second-order boundary value problem (on page 18) has aroused our interest in the following problem:

Given:	a real HILBERT space $(H, \langle \cdot, \cdot \rangle)$
	a continuous linear functional $\ell : H \longrightarrow \mathbb{R}$
Wanted:	an element $u \in H$ with
	$\langle u, v \rangle = \ell(v) \quad$ for every $v \in H$.

The central good news in this subsection is that this abstract problem always has a unique solution. In other words, every continuous linear functional on a HILBERT space can be represented as inner product with a fixed element which is even uniquely determined.

Theorem 4.1 (RIESZ' Representation Theorem)
Suppose $(H, \langle \cdot, \cdot \rangle)$ to be a real HILBERT space.

For every continuous linear functional $\ell : H \longrightarrow \mathbb{R}$, there exists a unique vector $\widetilde{u} \in H$ satisfying $\langle \widetilde{u}, v \rangle = \ell(v)$ for every $v \in H$.

Proof: It is based on essentially the same idea as DIRICHLET's principle in Proposition 2.2 (on page 5), i.e., minimizing an appropriate (possibly nonlinear) functional on H. Indeed, define

$$\Psi : H \longrightarrow R, \qquad u \longmapsto \frac{1}{2} \|u\|_H^2 - \ell(u)$$

with the norm $\|u\|_H := \sqrt{\langle u, u \rangle}$ on H related to its inner product $\langle \cdot, \cdot \rangle$.

Step 1: $\Psi : H \longrightarrow \mathbb{R},\, u \longmapsto \frac{1}{2} \cdot \langle u, u\rangle - \ell(u)$ is bounded from below.

Indeed, the assumed continuity of $\ell(\cdot)$ at $0_H \in H$ ensures a constant $\lambda \geq 0$ with $|\ell(v)| \leq \lambda\, \|v\|_H$ for every $v \in H$. Hence, we obtain for every $u \in H$

$$\begin{aligned} \Psi(u) &= \tfrac{1}{2} \cdot \langle u, u\rangle - \ell(u) \geq \tfrac{1}{2}\, \|u\|_H^2 - \lambda\, \|u\|_H \\ &\geq \inf_{r \in \mathbb{R}} \Big(\tfrac{1}{2}\, r^2 - \lambda\, r\Big) \geq -\tfrac{1}{2}\, \lambda^2 . \end{aligned}$$

From now on, let $(u_k)_{k\in\mathbb{N}}$ denote any minimizing sequence of Ψ in H, i.e., an arbitrary sequence in H satisfying

$$\Psi(u_k) \longrightarrow \inf_H \Psi > -\infty \qquad (k \longrightarrow \infty).$$

Step 2: $(u_k)_{k\in\mathbb{N}}$ is a CAUCHY sequence in $(H,\, \|\cdot\|_H)$,

Indeed, the key tool is the so-called *parallelogram equality* characterizing all normed linear spaces whose norm are induced by an inner product

$$\|\, u + v\,\|_H^2 + \|\, u - v\,\|_H^2 = 2 \cdot \Big(\|u\|_H^2 + \|v\|_H^2\Big)$$

for every $u, v \in H$ or, equivalently,

$$\Big\langle \tfrac{1}{2}\,(u+v),\ \tfrac{1}{2}\,(u+v)\Big\rangle + \Big\langle \tfrac{1}{2}\,(u-v),\ \tfrac{1}{2}\,(u-v)\Big\rangle = \tfrac{1}{2} \cdot \big(\langle u, u\rangle + \langle v, v\rangle\big).$$

(It is not difficult to verify since $\langle\cdot,\cdot\rangle$ is both bilinear and symmetric.) We obtain for any indices $j, k \in \mathbb{N}$

$$\begin{aligned} \Big\|\tfrac{1}{2}\,(u_j - u_k)\Big\|_H^2 &= \Big\langle \tfrac{1}{2}\,(u_j - u_k),\ \tfrac{1}{2}\,(u_j - u_k)\Big\rangle \\ &= \tfrac{1}{2} \cdot \big(\langle u_j, u_j\rangle + \langle u_k, u_k\rangle\big) - \Big\langle \tfrac{1}{2}\,(u_j + u_k),\ \tfrac{1}{2}\,(u_j + u_k)\Big\rangle \\ &= \Psi(u_j) + \ell(u_j) + \Psi(u_k) + \ell(u_k) - 2\Big(\Psi\Big(\tfrac{1}{2}\,(u_j + u_k)\Big) + \ell\Big(\tfrac{1}{2}\,(u_j + u_k)\Big)\Big) \\ &= \Psi(u_j) + \Psi(u_k) - 2 \cdot \Psi\Big(\tfrac{1}{2}\,(u_j + u_k)\Big) \\ &\leq \Psi(u_j) + \Psi(u_k) - 2 \cdot \inf_H \Psi. \end{aligned}$$

For any $\varepsilon > 0$ fixed arbitrarily, there is an index $k_\varepsilon \in \mathbb{N}$ with

$$\inf_H \Psi \leq \Psi(u_k) < \inf_H \Psi + \varepsilon \qquad \text{for all } k \geq k_\varepsilon.$$

Thus, $\Big\|\tfrac{1}{2}\,(u_j - u_k)\Big\|_H^2 < \varepsilon$ holds for any $j, k \geq k_\varepsilon$, i.e., $(u_k)_{k\in\mathbb{N}}$ is a CAUCHY sequence in H.

Step 3: $\Psi : H \longrightarrow \mathbb{R}$ attains its global minimum at $\widetilde{u} := \lim_{k\to\infty} u_k \in H$.

Indeed, both $\langle\cdot,\cdot\rangle$ and ℓ are continuous and so is Ψ. Thus, we conclude

$$\Psi(\widetilde{u}) = \Psi\Big(\lim_{k\to\infty} u_k\Big) = \lim_{k\to\infty} \Psi(u_k) = \inf_H \Psi .$$

Step 4: $\langle\widetilde{u},\ v\rangle = \ell(v)$ holds for every $v \in H$.

Indeed, for every $v \in H \setminus \{0_H\}$, the auxiliary function $\varphi_v : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}
\varphi_v(\sigma) &:= \Psi(\widetilde{u} + \sigma \cdot v) \\
&= \tfrac{1}{2} \cdot \langle \widetilde{u} + \sigma \cdot v,\ \widetilde{u} + \sigma \cdot v\rangle \qquad - \ell\,(\widetilde{u} + \sigma \cdot v) \\
&= \tfrac{1}{2} \cdot \|\widetilde{u}\|_H^2 + \sigma \cdot \langle\widetilde{u},\ v\rangle + \tfrac{1}{2}\,\sigma^2 \cdot \|v\|_H^2 - \ell\,(\widetilde{u}) - \sigma \cdot \ell(v) \\
&= \tfrac{\|v\|_H^2}{2}\,\sigma^2 + \Big(\langle\widetilde{u},\ v\rangle - \ell(v)\Big)\,\sigma + \tfrac{1}{2} \cdot \|\widetilde{u}\|_H^2 - \ell\,(\widetilde{u})
\end{aligned}$$

is a quadratic polynomial and so, it attains its global minimum at $\sigma = \frac{\langle\widetilde{u},\, v\rangle - \ell(v)}{\|v\|_H^2}$. Step 3, however, requires that this position is $\sigma = 0$. Thus we conclude

$$\langle\widetilde{u},\ v\rangle - \ell(v) = 0 .$$

Step 5: There is *at most* one $\widetilde{w} \in H$ satisfying $\langle\widetilde{w},\ v\rangle = \ell(v)$ for all $v \in H$.

Suppose that $\widetilde{w} \in H$ has this property. We conclude from step 4

$$\wedge \begin{cases} \langle\widetilde{u},\ v\rangle = \ell(v) & \text{for every } v \in H \\ \langle\widetilde{w},\ v\rangle = \ell(v) & \text{for every } v \in H \end{cases}$$

and thus, $\langle\widetilde{u} - \widetilde{w},\ v\rangle = 0$ is satisfied for every $v \in H$. For $v := \widetilde{u} - \widetilde{w} \in H$, in particular, we obtain

$$0 = \langle\widetilde{u} - \widetilde{w},\ \widetilde{u} - \widetilde{w}\rangle = \|\widetilde{u} - \widetilde{w}\|_H^2 \quad\Longrightarrow\quad \widetilde{u} = \widetilde{w}.$$

Hence, the limit $\widetilde{u} \in H$ (of an arbitrary minimizing sequence of $\Psi(\cdot)$) is always the unique solution to the representation problem related to $\ell(\cdot)$.

This completes the proof of RIESZ' Representation Theorem. □

Remark 4.2 Every vector u of a HILBERT space $(H, \langle\cdot,\cdot\rangle)$ induces a linear functional, namely $\ell_u : H \longrightarrow \mathbb{R},\ v \longmapsto \langle u, v\rangle$. It is even continuous because the CAUCHY–SCHWARZ inequality implies for every $v \in H$

$$|\ell_u(v)| \overset{\text{Def.}}{=} |\langle u, v\rangle| \leq \|u\|_H \|v\|_H .$$

This observation leads to a mapping from the real HILBERT space H to its so-called *dual space* H' (consisting of all continuous linear functionals $H \longrightarrow \mathbb{R}$)

$$H \longrightarrow H' \overset{\text{Def.}}{=} \mathrm{Lin}(H, \mathbb{R}), \qquad u \longmapsto \ell_u \overset{\text{Def.}}{=} \langle u, \cdot\,\rangle.$$

In short, this mapping is bijective – as an immediate consequence of RIESZ' Representation Theorem 4.1 (on page 33).

Furthermore, it is not difficult to verify that both this function and its inverse are linear. Next we focus on the aspect of continuity.

Definition 4.3 *Assume* $(H, \langle\cdot,\cdot\rangle)$ *to be a real* HILBERT *space and* $\ell : H \longrightarrow \mathbb{R}$ *a linear functional. The* (operator) norm *of* $\ell(\cdot)$ *is defined as*

$$\|\ell\|_{\mathrm{op}} := \sup\left\{ |\ell(v)| \;\middle|\; v \in H,\ \|v\|_H \leq 1\right\} \in [0, \infty[\cup \{\infty\} .$$

The following equivalence is a standard result in linear functional analysis (even on normed vector spaces), which we have already used in the proof of RIESZ' Representation Theorem:

Lemma 4.4 *A linear functional* $\ell : H \longrightarrow \mathbb{R}$ *on a real* HILBERT *space* $(H, \langle\cdot,\cdot\rangle)$ *is continuous on* H *if and only if* $\|\ell\|_{\mathrm{op}} < \infty$. □

Proposition 4.5 *Let* $(H, \langle\cdot,\cdot\rangle)$ *be a real* HILBERT *space. Suppose the continuous linear functional* $\ell : H \longrightarrow \mathbb{R}$ *to be represented by the vector* $\widetilde{u} \in H$ *in the sense that* $\langle\widetilde{u}, \cdot\,\rangle = \ell(\cdot)$ *in* H *(as in* RIESZ' *Representation Theorem 4.1). Then,* $\|\widetilde{u}\|_H = \|\ell\|_{\mathrm{op}}$.

Proof: *Step 1:* $\|\ell\|_{\mathrm{op}} \leq \|\widetilde{u}\|_H$

As in Remark 4.2, the CAUCHY–SCHWARZ inequality guarantees for all $v \in H$

$$|\ell(v)| = |\langle \widetilde{u}, v \rangle| \leq \|\widetilde{u}\|_H \, \|v\|_H$$

and thus, we obtain $\|\ell\|_{\mathrm{op}} \overset{\text{Def.}}{=} \sup \left\{ |\ell(v)| \;\middle|\; v \in H,\ \|v\|_H \leq 1 \right\} \leq \|\widetilde{u}\|_H$.

Step 2: $\|\widetilde{u}\|_H \leq \|\ell\|_{\mathrm{op}}$

If $\widetilde{u}$ is the null vector in H then the claim is trivial. Hence we are free to assume $\widetilde{u} \neq 0_H$ in addition. The characterizing criterion $\langle \widetilde{u}, v \rangle = \ell(v)$ for every $v \in H$ implies for $v := \widetilde{u}$, in particular,

$$\begin{aligned} \|\widetilde{u}\|_H^2 &= \langle \widetilde{u}, \widetilde{u} \rangle = \ell(\widetilde{u}) \leq \|\ell\|_{\mathrm{op}} \, \|\widetilde{u}\|_H \\ \Longrightarrow \quad & \|\widetilde{u}\|_H \leq \|\ell\|_{\mathrm{op}} \,. \end{aligned}$$

□

As a consequence, the linear mapping $H' \longrightarrow H$, $\ell \longmapsto \widetilde{u}$ is even an isometry and so is its inverse mentioned in Remark 4.2 (on page 35).

4.II $^{(*)}$ Extending the Representation to Symmetric Bilinear Forms: STAMPACCHIA's Theorem

RIESZ' Representation Theorem 4.1 (on page 33) proves to be a key tool indeed whenever dealing with continuous linear functionals on a real HILBERT space. In regard to boundary value problems of differential equations, however, we cannot expect that the generalised problem always relates the wanted element with a given linear functional in terms of the underlying inner product. As a slightly more general situation, a further symmetric bilinear form on H can be taken into consideration. This extension leads to the following problem:

Given:	a real HILBERT space $(H,\ \langle\cdot,\cdot\rangle)$
	a symmetric bilinear form $a : H \times H \longrightarrow \mathbb{R}$ (not necessarily $\langle\cdot,\cdot\rangle$)
	a continuous linear functional $\ell : H \longrightarrow \mathbb{R}$
Wanted:	an element $u \in H$ with
	$a(u,\ v) \ = \ \ell(v) \qquad$ for every $v \in H$.

The main goal in this subsection is to specify a sufficient condition on $a(\cdot,\cdot)$ so that both the existence and the uniqueness of the solution u can be concluded from RIESZ' Representation Theorem 4.1.

A rather obvious approach is based on the idea to supply the real linear space H with the bilinear form $a(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$. The standard definition of inner products, however, demands that $a : H \times H \longrightarrow \mathbb{R}$ is positive definite in addition, i.e., the following implication holds for every $v \in H$

$$a(v,v) \ = \ 0 \quad \Longrightarrow \quad v \ = \ 0_H \in H.$$

This condition suggests itself as appropriate answer to our question, but it is not sufficient ...

Indeed, even if $a(\cdot,\cdot)$ is an inner product on H (due to supplementary positivity) we still have to guarantee that H is complete with respect to its "new" norm

$$\|u\|_{H,\,a(\cdot,\cdot)} \ := \ \sqrt{a(u,u)} \qquad \text{instead of} \qquad \|u\|_H \ \overset{\text{Def.}}{=} \ \sqrt{\langle u,u\rangle}\ .$$

This observation motivates us to introduce a new condition on bilinear forms:

Definition 4.6 *Let $(H,\ \langle\cdot,\cdot\rangle)$ be a real* HILBERT *space. A bilinear form $a : H \times H \longrightarrow \mathbb{R}$ is called* coercive *if there is a constant $\alpha > 0$ with*

$$a(u,\ u) \ \ge \ \alpha\ \|u\|_H^2 \qquad \text{for every } u \in H.$$

In regard to norm (candidates) on the linear space H, this inequality implies $\|u\|_{H,\,a(\cdot,\cdot)} \ \ge \ \sqrt{\alpha}\ \ \|u\|_H$ for every vector $u \in H$, which we interpret as a first step towards the norm equivalence of $\|\cdot\|_{H,\,a(\cdot,\cdot)}$ and $\|\cdot\|_H$. The second step concerns another constant $\gamma > 0$ satisfying

$$\|u\|_{H,\,a(\cdot,\cdot)} \ \le \ \gamma\ \|u\|_H \qquad \text{for every } u \in H.$$

Indeed, then the completeness of $(H,\|\cdot\|_H)$ implies directly that $(H,\|\cdot\|_{H,\,a(\cdot,\cdot)})$ is also complete and so, $(H,\ a(\cdot,\cdot))$ is a real HILBERT space.

Similarly to linear functionals on H, upper estimates of bilinear forms are closely related with their continuity:

Lemma 4.7 *Let $a : H \times H \longrightarrow \mathbb{R}$ be a bilinear form on a real* HILBERT *space $(H,\ \langle\cdot,\cdot\rangle)$. Then the following statements are equivalent:*

(1.) *$a(\cdot,\cdot)$ is continuous (in $H \times H$).*

(2.) *$a(\cdot,\cdot)$ is continuous at $(0_H, 0_H) \in H \times H$.*

(3.) *There exists a constant $\beta \ge 0$ satisfying*

$$|\,a(v,w)\,| \ \le \ \beta\ \|v\|_H\ \|w\|_H \qquad \text{for every } v,w \in H.$$

Proof: "(1.) $\Longrightarrow$ (2.)" This is an obvious implication.

"(2.) $\Longrightarrow$ (3.)" $a(\cdot,\cdot)$ is assumed to be continuous at $(0_H, 0_H) \in H \times H$. Thus, there exists some $\delta > 0$ such that all $v, w \in H$ with $\|v\|_H \leq \delta$ and $\|w\|_H \leq \delta$ fulfil $\big|a(v,w)\big| \leq 1$. We conclude for arbitrary $v, w \in H \setminus \{0_H\}$

$$
\begin{aligned}
\big|a(v,w)\big| &= \Big|a\Big(\tfrac{\|v\|_H}{\delta}\,\tfrac{\delta}{\|v\|_H}\,v,\ \ \tfrac{\|w\|_H}{\delta}\,\tfrac{\delta}{\|w\|_H}\,w\Big)\Big| \\
&= \Big|\tfrac{\|v\|_H}{\delta}\cdot a\Big(\tfrac{\delta}{\|v\|_H}\,v,\ \tfrac{\|w\|_H}{\delta}\,\tfrac{\delta}{\|w\|_H}\,w\Big)\Big| \\
&= \tfrac{\|v\|_H}{\delta}\,\tfrac{\|w\|_H}{\delta}\cdot\Big|a\Big(\tfrac{\delta}{\|v\|_H}\,v,\ \tfrac{\delta}{\|w\|_H}\,w\Big)\Big| \ \leq\ \tfrac{\|v\|_H}{\delta}\,\tfrac{\|w\|_H}{\delta}\cdot\ 1\,.
\end{aligned}
$$

"(3.) $\Longrightarrow$ (1.)" By assumption, there exists a constant $\beta \geq 0$ with

$$
\big|\,a(v,w)\,\big| \ \leq\ \beta\,\|v\|_H\,\|w\|_H \qquad \text{for every } v, w \in H.
$$

Thus, we obtain for any vectors $v_1, v_2, w_1, w_2 \in H$

$$
\begin{aligned}
&\Big|a(v_1,w_1)\ -\ a(v_2,w_2)\Big| \\
&\leq\ \Big|a(v_1,w_1)\ -\ a(v_1,w_2)\Big|\ +\ \Big|a(v_1,w_2)\ -\ a(v_2,w_2)\Big| \\
&=\ \Big|a(v_1, w_1 - w_2)\Big| \qquad\qquad +\ \Big|a(v_1 - v_2,\ w_2)\Big| \\
&\leq\ \beta\,\|v_1\|_H\,\|w_1 - w_2\|_H \quad +\ \beta\,\|v_1 - v_2\|_H\,\|w_2\|_H\,.
\end{aligned}
$$

This inequality implies the continuity of $a(\cdot,\cdot)$ at any tuple $(v_1, w_1) \in H \times H$. □

Now we have completed the preparations for the main result of this subsection:

Theorem 4.8 (STAMPACCHIA **a.k.a.** LAX-MILGRAM **for symmetric bilin.forms)** *Assume the bilinear form $a : H \times H \longrightarrow \mathbb{R}$ on a real* HILBERT *space $(H, \langle\cdot,\cdot\rangle)$ to be symmetric, coercive and continuous, i.e.,*

(*i*) $a(u,v) = a(v,u)$ *for every $u, v \in H$,*

(*ii*) *there exists $\alpha > 0$ such that $a(u,u) \geq \alpha\ \|u\|_H^2$ holds for every $u \in H$,*

(*iii*) *there exists $\beta \geq 0$ with $\big|a(u,v)\big| \leq \beta\,\|u\|_H\,\|v\|_H$ for all $u, v \in H$.*

For every continuous linear functional $\ell : H \longrightarrow \mathbb{R}$, there exists a unique vector $\widetilde{u} \in H$ satisfying $a(\widetilde{u},\ v) = \ell(v)$ for every $v \in H$.

Proof: $a(\cdot,\cdot)$ is an inner product on the linear space H due to assumptions (i) , (ii) . Hypothesis (iii) implies

$$\alpha \ \|u\|_H^2 \ \leq \ a(u,u) \ \leq \ \beta \ \|u\|_H^2 \qquad \text{for every } u \in H$$

and so, the norms $\|\cdot\|_H$ and $\|\cdot\|_{H,\,a(\cdot,\cdot)} := \sqrt{a(\cdot,\cdot)}$ are equivalent.
Thus, $(H,\ a(\cdot,\cdot))$ is a real HILBERT space. Finally the claim is an immediate consequence of RIESZ' Representation Theorem 4.1 (on page 33). □

5 Analytical Foundations of GALERKIN's Method in a Separable HILBERT Space

Both RIESZ' Representation Theorem 4.1 (on page 33) and its extension, i.e., STAMPACCHIA's Theorem 4.8 (on page 40) represent two very useful tools in linear functional analysis. They specify sufficient conditions for existence and uniqueness of solutions to a rather broad class of problems in HILBERT spaces.

In regard to applications, however, they have a disadvantage in common: They are quite abstract and do not provide a constructive approach to the wanted solution – at first glance, at least.

For this reason we now focus on the question how to approximate the solution. The answer might also make a rather abstract impression because it considers just sequences (and subsequences) in a real HILBERT space. This approach, however, lays the foundations for finite element methods ... and so, it can lead us to concrete numerical results in the end. Admittedly, the very last step of implementing the approximations on a computer is beyond the scope of these lectures.

Suppose $(H, \langle\cdot,\cdot\rangle)$ to be a real HILBERT space and let any continuous linear functional $\ell : H \longrightarrow \mathbb{R}$ be given. Then RIESZ' Representation Theorem 4.1 guarantees both existence and uniqueness of the vector $\widetilde{u} \in H$ satisfying

$$\langle \widetilde{u}, v \rangle = \ell(v) \qquad \text{for every } v \in H.$$

Its proof reveals that $\widetilde{u}$ is the unique global minimum of the nonlinear functional

$$\Psi : \quad H \longrightarrow \mathbb{R}, \qquad u \longmapsto \tfrac{1}{2}\, \| u \|_H^2 - \ell(u)\,.$$

5.I Restricting the Problem to Finite-Dimensional Subspaces

Now the basic idea of the first step is to restrict all considerations to a finite-dimensional linear subspace H_m of H. ($m \in \mathbb{N}$ denotes the dimension of H.) As the key advantage of H_m over H, it is much simpler to find a vector $u_m \in H_m$ satisfying

$$\langle u_m,\ v\rangle \;=\; \ell(v) \qquad \text{for every } v \in H_m.$$

It is worth mentioning that the m-dimensional subspace $H_m \subset H$ occurs in two regards: H_m is not only the basic set in which the wanted vector u_m is selected, but it also restricts the vectors v in the variational equation (i.e., for which the functional $\ell(\cdot)$ is represented by u_m).

Let $\widehat{e}_1, \ldots, \widehat{e}_m$ form any orthonormal basis of H_m w.r.t. $\langle\cdot,\cdot\rangle$. Then every vector $v \in H_m$ has the unique representation

$$v \;=\; \langle v, \widehat{e}_1\rangle\ \widehat{e}_1 \;+\; \langle v, \widehat{e}_2\rangle\ \widehat{e}_2 \;+\; \ldots \;+\; \langle v, \widehat{e}_m\rangle\ \widehat{e}_m$$

and so, the linearity of ℓ leads to

$$\begin{aligned}
\ell(v) &= \ell\Big(\langle v, \widehat{e}_1\rangle\ \widehat{e}_1 + \langle v, \widehat{e}_2\rangle\ \widehat{e}_2 + \ldots + \langle v, \widehat{e}_m\rangle\ \widehat{e}_m\Big)\\
&= \langle v, \widehat{e}_1\rangle\ \ell(\widehat{e}_1) + \langle v, \widehat{e}_2\rangle\ \ell(\widehat{e}_2) + \ldots + \langle v, \widehat{e}_m\rangle\ \ell(\widehat{e}_m)\\
&= \langle v,\ \ell(\widehat{e}_1)\cdot\widehat{e}_1\rangle + \langle v,\ \ell(\widehat{e}_2)\cdot\widehat{e}_2\rangle + \ldots + \langle v,\ \ell(\widehat{e}_m)\cdot\widehat{e}_m\rangle\\
&= \Big\langle v,\ \ell(\widehat{e}_1)\cdot\widehat{e}_1 + \ell(\widehat{e}_2)\cdot\widehat{e}_2 + \ldots + \ell(\widehat{e}_m)\cdot\widehat{e}_m\Big\rangle,
\end{aligned}$$

i.e., the vector $u_m := \ell(\widehat{e}_1)\cdot\widehat{e}_1 + \ldots + \ell(\widehat{e}_m)\cdot\widehat{e}_m \in H_m$ fulfils

$$\ell(v) \;=\; \langle v,\ u_m\rangle \;=\; \langle u_m,\ v\rangle \qquad \text{for every } v \in H_m\,.$$

Finally this criterion characterizes $u_m \in H_m$ uniquely as the last step in the proof of RIESZ' Representation Theorem reveals (on page 35) when applied to the HILBERT space $(H_m, \langle\cdot,\cdot\rangle)$ and the restriction $\ell|_{H_m} : H_m \longrightarrow \mathbb{R}$.

5.II Asymptotic Features of the Approximative Solutions

The construction in the preceding subsection provides an "approximative" solution $u_m \in H_m$ for each m-dimensional linear subspace H_m of the real HILBERT space $(H, \langle\cdot,\cdot\rangle)$. It is uniquely determined by the condition

$$\langle u_m, v\rangle = \ell(v) \qquad \text{for every } v \in H_m.$$

Next we "increase" the dimension m of the subspace $H_m \subset H$. More rigorously speaking, we consider a sequence $(H_m)_{m\in\mathbb{N}}$ of linear subspaces of H with $\dim H_m = m$ for each $m \in \mathbb{N}$. It is related to a sequence $(u_m)_{m\in\mathbb{N}}$ of "approximative" solutions and, we hope that its asymptotic features lead to the wanted solution $\widetilde{u} \in H$ of

$$\langle \widetilde{u}, v\rangle = \ell(v) \qquad \text{for every } v \in H.$$

This method is likely to succeed in general only if every candidate for $\widetilde{u} \in H$ can be approximated by such a sequence (i.e., with its m-th member belonging to H_m for each $m \in \mathbb{N}$ respectively). Hence we require the additional feature

$$H = \overline{\bigcup_{m\in\mathbb{N}} H_m}\,.$$

Due to the finite dimension of each linear subspace H_m, it proves to be equivalent to a topological property of the underlying HILBERT space H:

Definition 5.1 *A normed vector space $(H, \|\cdot\|_H)$ is called* separable *if there exists a sequence $(\xi_k)_{k\in\mathbb{N}}$ with*

$$H = \overline{\left\{\xi_1, \xi_2, \xi_3, \ldots\right\}}$$

or, equivalently, if there exists a countable subset $M = \{\xi_1, ,\xi_2, \ldots\} \subset H$ such that every vector in H proves to be the limit of a sequence in M.

That is essentially all we need for an approximation method usually named after the Russian mathematician BORIS G. GALERKIN (1871 – 1945):

Proposition 5.2 (GALERKIN's method: Gist for inner product)
Suppose the real HILBERT *space* $(H, \langle\cdot,\cdot\rangle)$ *to be separable. Let* $\{\xi_1, \ ,\xi_2, \ldots\}$ *denote a countable dense subset of* H *and define*

$$H_m \ := \ \mathbb{R}\,\xi_1 \ + \ \mathbb{R}\,\xi_2 \ + \ \ldots \ + \ \mathbb{R}\,\xi_m \ \subset \ H \qquad \text{for each } m \in \mathbb{N}.$$

For any continuous linear functional $\ell : H \longrightarrow \mathbb{R}$*, the following statements hold:*

(*i*) *For each index* $m \in \mathbb{N}$*, there exists a unique vector* $u_m \in H_m$ *satisfying* $\langle u_m, v\rangle = \ell(v)$ *for every* $v \in H_m$.

(*ii*) *The resulting sequence* $(u_m)_{m\in\mathbb{N}}$ *converges to a vector* $\widetilde{u} \in H$.

(*iii*) *This limit* $\widetilde{u} \in H$ *fulfils* $\langle u, v\rangle = \ell(v)$ *for every* $v \in H$.

Proof: Statement (i) is an immediate consequence of RIESZ' Representation Theorem 4.1 (on page 33). Each vector $u_m \in H_m$ can even be constructed explicitly as discussed in § 5.I.

In regard to statements (ii), (iii), we rely on the alternative characterization of each vector $u_m \in H_m$. Indeed, consider the functional

$$\Psi : \quad H \ \longrightarrow \ \mathbb{R}, \qquad v \ \longmapsto \ \tfrac{1}{2}\,\langle v, v\rangle \ - \ \ell(v),$$

which is bounded from below. According to the proof of RIESZ' Representation Theorem 4.1, $u_m \in H_m$ is the unique minimizer of $\Psi|_{H_m}$.
We conclude indirectly that $(u_m)_{m\in\mathbb{N}}$ is a minimizing sequence of $\Psi(\cdot)$ in H. Indeed, the inclusion property $H_m \subset H_{m+1}$ (due to construction) implies

$$\Psi(u_m) \ = \ \min_{H_m} \ \Psi \ \geq \ \min_{H_{m+1}} \ \Psi \ = \ \Psi(u_{m+1})$$

for every index $m \in \mathbb{N}$, i.e., the real sequence $\big(\Psi(u_m)\big)_{m\in\mathbb{N}}$ is nonincreasing. Moreover, it is bounded from below by $\inf_H \ \Psi \ > \ -\infty$ and so, $\big(\Psi(u_m)\big)_{m\in\mathbb{N}}$ converges to its infimum.

Assume for a moment

$$\varepsilon \;:=\; \inf_{m \in \mathbb{N}} \; \Psi(u_m) \;-\; \inf_{H} \; \Psi \;\; > \; 0.$$

Then there exists a vector $w \in H$ satisfying $\inf_H \; \Psi \;\le\; \Psi(w) \;\le\; \inf_H \; \Psi + \frac{\varepsilon}{4}$.
Next we can always find an index $n \in \mathbb{N}$ with

$$\Big| \Psi(w) \;-\; \Psi(\xi_n) \Big| \;<\; \frac{\varepsilon}{4}$$

because $\Psi(\cdot)$ is continuous and $\{\xi_1, \, , \xi_2, \; \ldots \; \}$ is a dense subset of H. Hence, we conclude from $\xi_n \in H_n$

$$\begin{aligned}
\Psi(u_n) \;\;&=\;\; \min_{H_n} \; \Psi \qquad\quad \le \;\; \Psi(\xi_n) \\
&\le\;\; \Psi(w) + \frac{\varepsilon}{4} \;\;\; < \;\; \inf_{H} \; \Psi + \frac{\varepsilon}{4} \; + \; \frac{\varepsilon}{4} \\
&=\;\; \inf_{H} \; \Psi \; + \; \frac{\varepsilon}{2} \;\; = \;\; \inf_{m \in \mathbb{N}} \; \Psi(u_m) \; - \; \frac{\varepsilon}{2} \, .
\end{aligned}$$

This, however, is a contradiction and so, $(u_m)_{m \in \mathbb{N}}$ must be a minimizing sequence of Ψ in the sense that $\Psi(u_m) \longrightarrow \inf_H \; \Psi$ for $m \longrightarrow \infty$.

Finally, the proof of RIESZ' Representation Theorem 4.1 reveals the convergence of $(u_m)_{m \in \mathbb{N}}$ and that its limit $\widetilde{u} \in H$ is the unique vector satisfying the following equivalent conditions:

- $\Psi(\widetilde{u}) \;\;=\;\; \inf_H \; \Psi$
- $\langle \widetilde{u}, \; v \rangle \;\;=\;\; \ell(v)$ for every $v \in H$.

This completes the proof of statements (ii), (iii) here. □

Now we have laid the analytical foundations for approximating the wanted vector $\widetilde{u} \in H$ representing the continuous linear functional $\ell(\cdot)$ given.
The underlying convergence, however, is rather a qualitative feature so far. In regard to numerical implementations later on, it would be highly recommendable to estimate the error $\|\widetilde{u} - u_m\|_H$. This gap is not so difficult to bridge for GALERKIN's method – which we interpret as a further advantage of this approach:

Proposition 5.3 (Error estimate for GALERKIN's method)

Under the assumptions of preceding Proposition 5.2, the sequence $(u_m)_{m\in\mathbb{N}}$ *constructed there and its limit* $\widetilde{u} \in H$ *always satisfy the following inequality for every index* $m \in \mathbb{N}$

$$\|\widetilde{u} - u_m\|_H \;\leq\; \text{dist}(\widetilde{u},\, H_m)\,.$$

Proof: By construction, the vectors $u_m \in H_m \subset H$, $m \in \mathbb{N}$, and $\widetilde{u} \in H$ fulfil

$$\wedge \begin{cases} \langle u_m,\, v\rangle \;=\; \ell(v) & \text{for every } v \in H_m \subset H, \\ \langle\, \widetilde{u},\, v\rangle \;=\; \ell(v) & \text{for every } v \in H \end{cases}$$

and so, we obtain the so-called GALERKIN *orthogonality*

$$\langle \widetilde{u} - u_m,\; v\rangle \;=\; 0 \qquad \text{for every } v \in H_m \subset H.$$

It implies for every vector $v \in H_m$ (and index $m \in \mathbb{N}$)

$$\begin{aligned} \|\widetilde{u} - u_m\|_H^2 \;&\overset{\text{Def.}}{=}\; \langle \widetilde{u} - u_m,\; \widetilde{u} - u_m\rangle \\ &=\; \langle \widetilde{u} - u_m,\; \widetilde{u} - v \;+\; v - u_m\rangle \\ &=\; \langle \widetilde{u} - u_m,\; \widetilde{u} - v\rangle \;+\; \langle \widetilde{u} - u_m,\; v - u_m\rangle \\ &=\; \langle \widetilde{u} - u_m,\; \widetilde{u} - v\rangle \;+\; 0 \\ &\leq\; \|\widetilde{u} - u_m\|_H\; \|\widetilde{u} - v\|_H \end{aligned}$$

due to the CAUCHY–SCHWARZ inequality, i.e.,

$$\begin{aligned} & \|\widetilde{u} - u_m\|_H \;\leq\; \|\widetilde{u} - v\|_H \qquad \text{for every } v \in H_m \\ \Longrightarrow\qquad & \|\widetilde{u} - u_m\|_H \;\leq\; \inf_{v\in H_m} \|\widetilde{u} - v\|_H \;\overset{\text{Def.}}{=}\; \text{dist}(\widetilde{u},\, H_m). \end{aligned}$$

□

Admittedly, the distance of an unknown vector $\widetilde{u} \in H$ from a linear subspace $H_m \subset H$ is rather difficult to determine exactly. Hence, the established way-out in numerics is based on the notion to approximate the linear projection mapping $H \longrightarrow H_m$ by a linear auxiliary function $\varphi_m : H \longrightarrow H_m$ which is much easier to calculate numerically.

Then the estimate $\|\widetilde{u}\|_H = \|\ell\|_{\mathrm{op}}$ from Proposition 4.5 (on page 36) leads to the upper error bound

$$\begin{aligned}
\|\widetilde{u} - u_m\|_H &\leq \mathrm{dist}(\widetilde{u}, H_m) \\
&\leq \|\widetilde{u} - \varphi_m(\widetilde{u})\|_H \\
&\leq \sup\left\{ \|w - \varphi_m(w)\|_H \;\middle|\; w \in H,\ \|w\|_H \leq \|\ell\|_{\mathrm{op}} \right\} \\
&= \|\mathbb{1}_H - \varphi_m\|_{\mathrm{op}} \ \ \|\ell\|_{\mathrm{op}} .
\end{aligned}$$

This inequality gives us a first flavour of how the sophisticated choice of both linear subspaces $H_m \subset H$ $(m \in \mathbb{N})$ and the so-called "interpolation mapping" $\varphi_m : H \longrightarrow H_m$ has an essential influence on the speed of convergence of GALERKIN's method(s).

5.III Application to Elliptic Partial Differential Equations

The way to GALERKIN's method in a separable HILBERT space was motivated by the concrete example of POISSON's equation with zero DIRICHLET boundary conditions

$$\wedge \begin{cases} \Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now we summarize an adequate choice for the HILBERT space in this example: For a nonempty open bounded set $\Omega \subset \mathbb{R}^n$ and

$$W_0^{1,2}(\Omega) \stackrel{\text{Def.}}{=} \overline{C_c^1(\Omega)} \quad (\text{w.r.t. } \|\cdot\|_{W^{1,2}(\Omega)}),$$

$$\langle u, v\rangle_{W_0^{1,2}(\Omega)} \stackrel{\text{Def.}}{=} \int_\Omega \langle \nabla u(x),\ \nabla v(x)\rangle_{\mathbb{R}^n}\ d\mathcal{L}^n x,$$

the tuple $\left(W_0^{1,2}(\Omega),\ \langle\cdot,\cdot\rangle_{W_0^{1,2}(\Omega)}\right)$ is a real HILBERT space due to Corollary 3.12 (on page 32). For any $f \in L^2(\Omega)$ given, RIESZ' Representation Theorem 4.1 (on page 33) is applied to the linear functional

$$\ell: \quad W_0^{1,2}(\Omega) \longrightarrow \mathbb{R}, \qquad \varphi \longmapsto -\int_\Omega f(x)\cdot\varphi(x)\ d\mathcal{L}^n x$$

whose continuity results from HÖLDER's and POINCARÉ's inequalities. It leads directly to the existence and uniqueness of weak solutions:

Proposition 5.4 *Let the nonempty set $\Omega \subset \mathbb{R}^n$ be bounded and open. Then for every function $f \in L^2(\Omega)$, there exists a unique weak solution $u : \Omega \longrightarrow \overline{\mathbb{R}}$ to the boundary value problem*

$$\wedge \begin{cases} \Delta u = f & \textit{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \textit{on } \partial\Omega \end{cases}$$

in the sense of Definition 3.9, i.e., a unique function $u \in W_0^{1,2}(\Omega)$ satisfies

$$\int_\Omega \langle \nabla u(x),\ \nabla\varphi(x)\rangle_{\mathbb{R}^n}\ d\mathcal{L}^n x = -\int_\Omega f(x)\cdot\varphi(x)\ d\mathcal{L}^n x$$

for every test function $\varphi \in W_0^{1,2}(\Omega)$.

Stronger assumptions about $f(\cdot)$ and $\Omega \subset \mathbb{R}^n$ are sufficient for this $u(\cdot)$ to be a solution in the classical sense, i.e., $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ with $\Delta u = f$ and $u|_{\partial\Omega} = 0$. These details, however, are beyond the scope of this course.

It is essential that this approach works for a much larger class of partial differential equations. We can use STAMPACCHIA's Theorem 4.8 about symmetric bilinear forms (on page 40) instead of RIESZ' Representation Theorem.
It merely remains to ensure that the underlying bilinear form is coercive in the sense of Definition 4.6 (on page 39). This condition leads to so-called *elliptic* partial differential equations.

Proposition 5.5 (Weak solution to elliptic partial differential equation)
Let the nonempty set $\Omega \subset \mathbb{R}^n$ be bounded and open. Suppose for $a_{jk}, f : \Omega \longrightarrow \mathbb{R}$ $(j, k \in \{1, \ldots, n\})$

(*i*) $a_{jk} \in L^\infty(\Omega)$ *for every* $j, k \in \{1, \ldots, n\}$,

(*ii*) *the matrix* $(a_{jk}(x))_{1 \leq j,k \leq n}$ *is symmetric for every* $x \in \Omega$,

(*iii*) *there exists a constant* $\lambda > 0$ *in regard to the* uniform ellipticity condition
$$\sum_{j,k=1}^{n} a_{jk}(x)\ \xi_j\ \xi_k \quad \geq \quad \lambda\ \|\xi\|^2 \qquad \text{for every } \xi \in \mathbb{R}^n,\ x \in \Omega,$$

(*iv*) $f \in L^2(\Omega)$.

Then there exists a unique weak solution $u : \Omega \longrightarrow \mathbb{R}$ to the elliptic differential equation "in divergence form" with zero DIRICHLET *boundary condition*
$$\wedge \begin{cases} -\displaystyle\sum_{j,k=1}^{n} \partial_k\Big(a_{jk}(\cdot)\ \partial_j u\Big) \;=\; f & \text{in } \Omega \subset \mathbb{R}^n \\ \qquad\qquad\qquad\qquad u \;=\; 0 & \text{on } \partial\Omega, \end{cases}$$

i.e., a unique function $u \in W_0^{1,2}(\Omega)$ satisfies for every test function $\varphi \in W_0^{1,2}(\Omega)$
$$\int_\Omega \sum_{j,k=1}^{n} a_{jk}(x) \cdot \partial_j u(x) \cdot \partial_k \varphi(x)\ d\mathcal{L}^n x \;=\; \int_\Omega f(x) \cdot \varphi(x)\ d\mathcal{L}^n x.$$

Proof: The claim results directly from STAMPACCHIA's Theorem 4.8 (page 40). Consider

$$\begin{aligned} H &:= W_0^{1,2}(\Omega), \\ \langle u, v\rangle &:= \langle u, v\rangle_{W_0^{1,2}(\Omega)} \overset{(3.11)}{=} \int_\Omega \langle \nabla u(x),\ \nabla v(x)\rangle_{\mathbb{R}^n}\ d\mathcal{L}^n x, \\ a(u,v) &:= \int_\Omega \sum_{j,k=1}^n a_{jk}(x)\cdot \partial_j u(x)\cdot \partial_k v(x)\ d\mathcal{L}^n x, \\ \ell(v) &:= \int_\Omega f(x)\cdot v(x)\ d\mathcal{L}^n x. \end{aligned}$$

The bilinear form $a(\cdot,\cdot)$ is symmetric due to the symmetry of each coefficient matrix $(a_{jk}(x))_{1\le j,k\le n} \in \mathbb{R}^{n\times n}$ for $x \in \Omega$.
Furthermore we conclude from assumption (i), HÖLDER's inequality as well as POINCARÉ's inequality (Proposition 3.10 on page 30)

$$\begin{aligned} |a(u,v)| &\le \int_\Omega \Big| \sum_{j,k=1}^n a_{jk}(x)\cdot \partial_j u(x)\cdot \partial_k v(x)\Big|\ d\mathcal{L}^n x \\ &\le \sum_{j,k=1}^n \int_\Omega \Big| a_{jk}(x)\cdot \partial_j u(x)\cdot \partial_k v(x)\Big|\ d\mathcal{L}^n x \\ &\le \sum_{j,k=1}^n \|a_{jk}\|_{L^\infty(\Omega)} \cdot \int_\Omega \Big|\partial_j u(x)\cdot \partial_k v(x)\Big|\ d\mathcal{L}^n x \\ &\le \sum_{j,k=1}^n \|a_{jk}\|_{L^\infty(\Omega)} \cdot \|\partial_j u\|_{L^2(\Omega)}\ \ \|\partial_k v(x)\|_{L^2(\Omega)} \\ &\le \sum_{j,k=1}^n \|a_{jk}\|_{L^\infty(\Omega)} \cdot \|u\|_{W_0^{1,2}(\Omega)}\ \ \|v\|_{W_0^{1,2}(\Omega)}, \end{aligned}$$

i.e., $a(\cdot,\cdot)$ is continuous w.r.t. the norm $\|\cdot\|_{W_0^{1,2}(\Omega)} \overset{\text{Def.}}{=} \sqrt{\langle\cdot,\cdot\rangle_{W_0^{1,2}(\Omega)}}$.

Finally, the uniform ellipticity condition in hypothesis (iii) implies that $a(\cdot,\cdot)$ is coercive. Indeed, we obtain for any $u \in W_0^{1,2}(\Omega)$

$$\begin{aligned} a(u,u) &= \int_\Omega \sum_{j,k=1}^n a_{jk}(x)\cdot \partial_j u(x)\cdot \partial_k u(x)\ d\mathcal{L}^n x \\ &\ge \int_\Omega \lambda\ \|\nabla u(x)\|^2\ d\mathcal{L}^n x \quad = \quad \lambda\ \|u\|^2_{W_0^{1,2}(\Omega)}. \end{aligned}$$ □

References

Lecture notes

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